The golden mean: The risk mitigating effect of combining tournament rewards with high-powered incentives *

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Abstract

The welfare of financial professionals frequently depends on both tournament rewards for relative performance—fund inflows based on fund ranking, promotions based on peer comparisons—and compensation-based absolute performance rewards—bonus payments for meeting accounting targets, hedge-fund performance fees. It is well known that both high-powered compensation and relative-performance rewards engender risk taking. In this paper, we show that these two sources of risk taking, rank and bonus compensation, mollify the risk-taking incentives produced by the other: high-powered compensation mitigates rank-motivated risk taking. Rank rewards mitigate compensation-motivated risk taking.

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1 Introduction

The welfare of financial agents frequently depends on rewards based on absolute performance targets, e.g., bonus payments to bankers and CEO's, payments to hedge-fund managers based on attaining "high-water marks." At the same time, financial agents' welfare also frequently depends on the ranking of their performance relative to their peers: mutual fund and hedge fund managers compete for rankings, a key determinant of fund inflows and thus fund manager compensation (Ma et al., 2019). Executives compete for promotions based on relative performance (Kini and Williams, 2012). In fact, an estimated 20% of large US firms use "forced ranking systems" that base the promotion and termination of managers entirely on relative performance (Bates, 2003). Ranking concerns even drive the risk-taking decisions of financial professionals when ranking is anonymous (and thus divorced from status) and has no effect on pecuniary rewards (Kirchler et al., 2018).

An extensive body of theoretical and empirical research suggests that high-powered absolute performance rewards lead to *ruin risk-taking* (also known as "gambling on resurrection"), i.e., choosing portfolios or investment projects that entail a significant probability of ruin but promise large upside returns (Rose-Ackerman, 1991; Palomino and Prat, 2003). For example, fund managers who accept ruin risk by writing naked out-of-the-money puts; CEOs who "bet the company" on speculative R&D projects, increase operating, or financial leverage. A smaller but substantial body of research suggests that competition for rank rewards also leads to ruin risk taking by weaker competitors (Hillman and Samet, 1987; Hillman and Riley, 1989).

What is less well understood is the risk-taking behavior of managers who receive a mixture of absolute and relative performance rewards. This paper shows that when rank and bonus are combined, each type of reward substantially mollifies the risk taking incentives produced by the other type. Managers incentivized by both rank rewards and high-powered performance rewards are willing to accept substantially less ruin risk than predicted by their compensation contracts or tournament incentives considered in isolation.

We develop these results in a model in which two managers with unequal ability compete for *performance* rewards.[1] Performance is a random variable. For example, hedge-fund manager performance might consist of the returns or net asset values. CEO performance might consist of quarterly revenue or profits.

Unequal ability is modeled by a *capacity* constraint, an upper bound on expected performance that is not the same for the two managers. This specification is consistent with portfolio construction in complete markets where agents can attain any payoff pattern that they can afford and the cost of portfolios equals their discounted expected value under a risk-neutral probability measure. As we show in Section 5.1, our analysis can be extended

[1] In Section 5.4.2 we show that the tradeoffs which drive our results extend to competitions among more than two managers.

to non-linear constraints on performance capacity, constraints which can ensure that high risk "asset substitution" strategies lower expected performance.

Absolute performance rewards are modeled as bonus payment received if and only if performance exceeds a threshold level. This specification, although stylized, closely tracks many standard incentive schemes, e.g., 80/120 bonus plans which are, by far, the most common bonus compensation schemes for managers in U.S. corporations (Murphy, 1999).[2] In the baseline model, we consider bonus targets that are not so low that managers can capture the bonus reward with certainty. Thus, in the absence of rank rewards, bonus rewards motivate managerial risk taking. In Section 5.2.2, we extend our analysis to consider option-based compensation and show that, although the modeling of option compensation is more complex, the qualitative features of risk taking under option compensation are quite similar to risk taking under bonus compensation.

In addition to bonus rewards, managers also receive *rank rewards* based on relative performance: they earn a rank reward whenever their performance tops the performance of their rival. Thus, abstracting from bonus rewards, our model is a risk-taking contest model quite similar to other models of rank competitions in financial markets (Seel and Strack, 2013; Strack, 2016), social status contests (Robson, 1992; Becker et al., 2005; Ray and Robson, 2012) and political contests (Myerson, 1993).

Our framework models mixed (rank and absolute performance) rewards by concatenating standard models of risk taking under bonus and rank-based reward. Combining absolute and relative performance rewards produces a fairly surprising result: introducing rank-rewards into competitions for absolute performance rewards always reduces ruin risk taking. At the same time, mixed rewards also always engender less ruin risk taking than competitions for rank rewards. Our analysis has many other implications for fund manager and CEO competitions, detailed in Section 6, which are by no means obvious in the absence of a formal model of mixed relative and absolute performance rewards.

The effects we identify arise because, although both rank competition and bonus competition encourage risk taking, optimal risk-taking strategies under bonus and rank competition are very different. Thus, when rewards are mixed, efficient risk taking in pursuit of bonus rewards has a significant opportunity cost—inefficient rank competition—and vice versa. These opportunity costs curtail extreme risk taking in pursuit of either rank or bonus rewards.

When performance targets are challenging and managers compete only for absolute performance rewards, optimal performance strategies involve concentrating performance by targeting upside performance levels and accepting some ruin risk. However, if managers also receive rank rewards, such targeting is costly in terms of lost rank-rewards. For this reason,

[2]Under an 80/120 bonus scheme, the firm sets a performance target and associated maximum bonus. If the manager's performance equals or exceeds 120% of the target, the manager receives the maximum bonus; if performance falls short of 80% of the target, no payment is made to the manager; between 80% and 120% of the target, the bonus payment increases linearly in performance.

once rank rewards are introduced, strong managers, who have the capacity to compete for rank dominance without accepting ruin risk, eschew performance strategies involving ruin risk and spread out their performance.[3] Weak managers, in order to compete with strong managers for rank dominance, also spread out their performance, and thereby reduce their ruin risk taking. Hence, the introduction of rank-based incentives reduces ruin risk taking.

However, ruin risk taking is not uniformly decreasing in the strength of rank incentives. Once rank-incentives are introduced, the ruin risk that weak managers must accept to compete for ranking with strong managers is proportional to the rank-competition efficiency of strong managers' performance strategies. Performance strategies that are efficient for capturing absolute performance rewards are inefficient for capturing relative-performance rewards. As the strength of rank incentives increase, strong managers' strategies become more rank-competition efficient. The increase in the rank efficiency of strong managers' strategies increases the ruin risk that weak managers must accept.

Thus, ruin risk taking is lowest when managers face mixed rank and absolute performance incentives. In fact, as detailed in Section 4.1.1, mixed incentive packages can be constructed that virtually eliminate ruin risk taking. Moreover, because of the incentive to spread out performance engendered by rank incentives, managerial performance under bonus incentives is never less risky (in the sense of second-order stochastic dominance) than performance under mixed incentives and, under fairly non-restrictive conditions, performance under mixed incentives is less risky, in the sense of second or third-order stochastic dominance, than performance under bonus incentives.

Contributions and related literature

An economic phenomenon modeled in this paper—the effects of relative rewards on managerial behavior—has been extensively analyzed in two wide streams of empirical finance research: risk-taking by fund managers (e.g., Kempf and Ruenzi, 2008; Chevalier and Ellison, 1997), and risk taking by CEOs (e.g., Coles et al., 2017; Kini and Williams, 2012; Kale et al., 2009). In fact, although most of the theory of risk taking in rank competitions has been developed in quite general settings in the economics literature, with the exception of sporting contests, almost all of the empirical research on contest risk taking and relative performance rewards has focused on financial agents, i.e., fund managers, top executives, or CEOs.

The objective of our analysis is the same as the objective of many other financial economics papers, e.g., Green (1984), Martynova and Perotti (2018), and Carpenter (2000): characterize the effect of given reward structures on managerial behavior rather than characterize optimal reward structures. Thus, our results do not depend on the origins of rank rewards or the

[3]Technically, "spread out performance" means that the Lebesgue measure of the support of the managers' performance distribution increases.

motivations for offering absolute performance compensation. As the literature discussed above notes, performance rewards, in practice, are generated by a variety of forces some of which are outside of the control of firms (e.g., career concerns).

The unique, to our knowledge, feature of our analysis is that it models (a) the risk-taking incentives of (b) agents who receive mixed relative and absolute performance rewards. A number of researchers have modeled the risk-taking incentives produced by absolute performance rewards (e.g., Green, 1984; Carpenter, 2000). Others modeled the effect of relative performance rewards on risk taking (e.g., Hvide, 2002; Strack, 2016). Kräkel and Schöttner (2008) consider the effect of mixed promotion (relative performance) and compensation (absolute performance) rewards on effort incentives. They assume that promotion is based on a reduced-form Tullock contest success function that maps deterministic effort levels into promotion probabilities. Because the distribution of agent performance is not explicitly modeled, this framework cannot model strategic risk taking.[4]

We develop our results within the risk-taking contest framework. This framework modifies the zero-noise "all-pay auction" rank-competition framework by limiting contestant performance through a capacity constraint rather than through bid or effort costs. The all-pay auction framework (e.g., Olszewski and Siegel, 2016; Siegel, 2009) has been extensively analyzed and is probably the most widely used framework for modeling rank competition in economics research. The virtue of all-pay and risk-taking contest models, from our perspective, is that these frameworks permit agents to choose optimal risk-taking strategies, which need not correspond to standard symmetric text-book statistical distributions. As we show in Section 5.1, our results do not depend on managers' risk-choice sets including all risk distributions that satisfy an expectational constraint. However, they do depend on managers being able to vary the shape of the risk distribution in addition to its scale. As discussed in more detail in Section 5.1, we think that this is an advantage, not a limitation, of our analysis.

2 Framework

In this section, we develop our baseline model of mixed competitions: there are two risk-neutral managers: a strong manager, S, and a weak manager, W. Managers choose random variables, X_i, with distributions F_i, $i = S, W$. The random variables, X_S and X_W, are independent. Thus, the managers' performance can be completely described by the marginal distributions, F_S and F_W of X_S and X_W. We will call F_i the manager i's *performance distribution.* We will call an arbitrary realization of the random variables X_i, $i = S, W$, *performance*, and represent performance with x.

[4]Ekinci et al. (2019) also consider combined effect of promotions and bonus compensation on effort and pay dispersion. However, in their analysis, the number of agents promoted is not fixed by a promotion quota, i.e., forced ranking system. Thus, in their model, promotion is not determined by rank competition.

Managers receive *rewards* based on performance. The rewards the managers receive are determined by their absolute and relative performance. The absolute performance reward takes the form of a bonus payment $B > 0$ received whenever a manager's performance weakly exceeds a bonus threshold, represented by θ. The relative performance reward consists of a *rank reward*, $R > 0$, that a manager receives if the manager's performance exceeds the performance of her rival. In the event of tied performance, the rank reward is split equally between the two managers.

The feasible set of performance distributions for manager i, $i = S, W$, consists of all performance distributions whose supports are contained in the non-negative real line and whose expectation is no greater than μ_i, $i = S, W$. μ_i thus represents the manager's *capacity*. We assume that bonus threshold, θ, is greater than the strong manager's capacity, μ_S, and that the strong manager has greater capacity than the weak manager, i.e., $\mu_S > \mu_W$.[5]

The sequence of actions in the competition is as follows: Managers pick performance distributions. A random draw from each distribution, x, a performance, is selected which determines absolute and relative performance rewards for the given draw. The managers' payoffs, i.e., utilities, are given by their expected rewards. Managers choose strategies, i.e., performance distributions, with the aim of maximizing their payoffs.

Our model differs from models of risk taking that extend the Lazear and Rosen (1981) tournament model to include risk taking (e.g., Hvide, 2002; Coles et al., 2020). In these models, contestants choose an effort level, which corresponds to μ in our model, and a scale parameter, σ. Contestant performance equals $\mu + \sigma\,\tilde{z}$ where $\tilde{z}$ is a symmetrically distributed zero-mean (typically Normally distributed) noise term. Risk taking in this framework reduces to the choice of the risk-scale parameter, σ. The "shape" of contestants' performance distributions, e.g., their skewness or kurtosis, is exogenously fixed. In contrast, as we show in the next section, the ability of managers to vary the shape of their performance distributions is a key feature of our analysis.

2.1 Example

In this section, we develop a numerical example that illustrates the logic behind our subsequent formal analysis. All of the analysis in this section is heuristic and elides some of the more subtle questions we formally address later in the paper. We derive equilibrium performance strategies in a *bonus competition*, where managers receive only absolute performance, and a *rank competition*, where managers receive only relative performance rewards. By comparing the nature of optimal risk-taking strategies in these two settings, we develop

[5] This assumption rules out symmetric capacity, $\mu_S = \mu_W$. The symmetric capacity case is quite easy to analyze. However, in the absence of strength asymmetry, rank competitions do not lead to ruin risk taking, a central focus of this paper. Hence, we do not think it is worthwhile to consider the symmetric case in our analysis. In Section 5.3 we consider bonus thresholds so low that managers can attain the bonus without taking any risks.

a number of observations about how the introduction of rank rewards is likely to affect the strategic behavior of managers in *mixed competitions*, where managers receive both absolute and relative performance rewards. The parameters assumed in this example are presented in Table 1.

S-capacity	W-capacity	Bonus threshold
$\mu_S = 4.125$	$\mu_W = 1.900$	$\theta = 7.000$

Table 1: Example parameters

2.1.1 Bonus competition

First, consider a bonus competition in which the bonus reward B, equals 1, and the rank reward, R, equals 0. In this competition, optimal performance distributions for S and W are quite easy to compute: given the capacity constraint, neither manager can submit performance that always weakly exceeds the bonus threshold, θ. Thus, both managers will choose performance distributions that maximize the probability of attaining θ performance subject to their capacity constraints. Hence, the managers will adopt a "bang-bang" strategy of placing all probability weight on 0 or θ. Let p_i^{bns} represent the probability of a performance of θ for a manager of type $i = S, W$. The capacity constraint is clearly binding. Thus, optimal performance strategies satisfy

$$(1 - p_i^{\text{bns}})\, 0 + p_i^{\text{bns}}\, \theta = \mu_i, \quad i = S, W.$$

Using the parameters given in Table 1, we see that $p_S^{\text{bns}} = {}^{33}\!/_{56} \approx 0.59$ and $p_W^{\text{bns}} = {}^{19}\!/_{70} \approx 0.27$. Consequently, optimal performance strategies under bonus competition involve *targeting the bonus*, i.e., placing probability mass at the bonus threshold. Because the managers must satisfy their capacity constraints, they cannot place all mass on the bonus threshold and thus they must accept some ruin risk, i.e., some probability that performance will equal 0. Finally, note that the payoffs of the two managers equal $p_i^{\text{bns}}\, B$, where $p_i^{\text{bns}} = \mu_i/\theta$. Thus, the marginal gain from increased capacity, μ_i, is B/θ for both managers.

There is a natural interpretation of the performance strategies we have just identified in the fund-manager competition context. In a complete and perfect financial market, the cost of any portfolio equals its expected value under the risk-adjusted market pricing measure. When hedge-fund managers have the ability to generate "alpha," i.e., portfolio value exceeding the cost of their portfolios, managerial performance represents dynamic trading strategies that allocate alpha across states of nature. So, at least roughly, the expectational constraint appears to fit hedge-fund manager competitions, at least when manager's trading strategies are not constrained by value-at-risk (VAR) restrictions and markets are fairly complete.[6] In the language of applied finance research, bonus targeting can be interpreted as buying lottery tickets that promise performance sufficient to capture the bonus; ruin risk taking can be interpreted as selling portfolio insurance (or puts) (Ilmanen, 2012). Under this interpretation, optimal risk taking strategies in bonus competition can be thought of as funding the purchase of lottery tickets though selling portfolio insurance.[7]

Expectational constraints have also been used to model the feasible set of managerial actions for corporate financial managers and venture capitalists (Admati and Pfleiderer, 1994; Ravid and Spiegel, 1997). However, in the context of corporate risk taking, i.e., risk shifting/asset substitution, most analysis has focused on situations in which risk taking lowers value. Thus, the fit between the risk-shifting setting and our expectational constraint is imperfect.[8] However, in Section 5.1, we will show that our results under expectational constraints are largely preserved by non-linear constraints on performance that impose a risk-return tradeoff. In fact, we will establish an isomorphism between competitions with expectational constraints and competitions that impose non-linear constraints on performance.

2.1.2 Introducing rank-based rewards

Suppose we hold the bonus reward, B, constant and introduce a positive rank reward, $R > 0$. The insight about rank contests that is key to our subsequent analysis is that buying lottery tickets is a terribly inefficient strategy for winning rank competitions. To see this, suppose that W chooses a performance distribution of the following form: with probability p_ϵ submit performance $X_W = \theta + \epsilon$, where $\epsilon > 0$ and $\epsilon \approx 0$ and submit performance $X_W = \epsilon$ with probability $1 - p_\epsilon$, where p_ϵ is fixed to satisfy the capacity constraint, $p_\epsilon\,(\theta + \epsilon) + (1 - p_\epsilon)\,\epsilon = \mu_W$.

If this distribution is used by W, and S uses her equilibrium bonus competition distribution, the performance of W will always top the performance of S when $X_W = \theta + \epsilon$ and will top S's performance with probability $1 - p_S^{\text{bns}}$ when $X_W = \epsilon$. Noting that $p_S^{\text{bns}} = 33/56$, we see that W's probability of eking out a rank-competition victory over S is given by

$$p_\epsilon + (1 - p_\epsilon)(1 - p_S^{\text{bns}}) = \frac{1}{56}(33\,p_\epsilon + 23).$$

Because $\epsilon \approx 0$, W's capacity constraint can be satisfied with $p_\epsilon \approx p_W^{\text{bns}} = \mu_W/\theta = 33/56$. Thus, W, with one-third the capacity of S, can attain a probability of winning the rank competition arbitrarily close to $\frac{1}{56}(33\,\frac{33}{56} + 23) \approx 0.57 > 0.50$, without materially lowering his probability of receiving the bonus reward.

[6] For an example of risk-taking models like ours used to model risk-taking by portfolio managers in complete markets, see Strack (2016).

[7] Chan et al. (2005) detail how hedge funds construct these "selling insurance" portfolio strategies which lead to either liquidation or very high returns, and explore the systemic costs of such strategies.

[8] Leland and Toft (1996), for example, formally model managerial ruin (i.e. bankruptcy) risk taking and its attendant costs. The idea that financial contracting can motivate costly ruin risk taking is much older and can be traced back at least to Jensen and Meckling (1976).

The problem with S's bonus competition strategy is its predictability: if W knows that S will devote all of her performance capacity to placing mass on a single performance level, θ, then W can "just top" S by submitting performance slightly greater than θ. Just topping θ requires using only negligibly more capacity than targeting θ, and thus must be balanced by only a negligible increase in ruin risk. At the same time, just topping yields a huge increase in W's probability of winning the rank reward. Once, rank rewards are introduced, the managers reduce the predictability of their performance strategies.

Note that reduced predictability is not synonymous with increased variance, or "risk" in the Rothschild and Stiglitz (1970) sense: If I know that my car keys are either in the nightstand drawer in my London flat or the nightstand drawer in my New Zealand vacation home, the variance of the location of my car keys is enormous. However, I can make a prediction about my key's location, to within one meter, that is bound to be right at least half of the time. In contrast, if all that I know is that I drop my car keys somewhere during a one kilometer walk down the high street, the variance of the location of the keys is relatively tiny, yet any prediction that I make about their location (to within one meter) has only 1/1000 chance of being correct.[9] To decrease predictability, S needs *spread out* her performance, i.e., decrease the probability weight placed on any interval of performance relative to the interval's length.

2.1.3 Rank competition

In order to see how much rank rewards can spread out the supports of the managers' equilibrium performance distributions, consider a rank competition: set the bonus reward to $B = 0$ and the rank reward to $R = 1$. In this setting, the competition reduces to a standard risk-taking contest game. The equilibria of such games have been extensively analyzed in the literature (e.g., Xiao, 2016; Hillman and Riley, 1989). The equilibrium calls for S to randomize uniformly over 0 to $2\,\mu_S = 8.25$, and W to submit a performance of 0 with probability $p_W^{\mathrm{rnk}}(0) = 1 - {}^{\mu_W}\!/_{\mu_S} = {}^{89}\!/_{165} \approx 0.54$ and, with probability $1 - p_W^{\mathrm{rnk}}(0) = {}^{\mu_W}\!/_{\mu_S} = {}^{76}\!/_{165} \approx 0.46$, submit performance distributed Uniform$[0, 8.25]$. Thus, under the capital markets interpretation, S does not sell insurance or buy lotter tickets. Rather S simply spreads out performance to avoid providing W with specific performance levels that W can profitably just top. W sells sufficient insurance to ensure that, whenever the insurance contract is "out of the money," he can shadow S's performance; in the event the insurance contract is in the money, W is ruined.

Verifying this equilibrium is straightforward. First, consider W: given the uniform distribution played by S, all distributions over $[0, 8.25]$ featuring the same expected performance

[9] It is possible to formalize "more or less predictable" using a partial order of random variables called the "uncertainty order" and developed in Marshall et al. (Chapt 17.F 2011). For absolutely continuous distribution functions, greater uncertainty in the uncertainty order implies, but is not implied by, greater entropy. Yang (2020) shows that entropy is useful for characterizing information production costs in a security design settings.

produce the same payoff. Performance in excess of 8.25 produces the same probability of winning the rank reward as 8.25, and uses up more capacity. A simple calculation shows that W's performance distribution satisfies his capacity constraint.

Similarly, for S, because W is randomizing between a point mass at 0 and a uniform distribution over $[0, 8.25]$, all performance distributions placing no probability mass at 0, supported by $[0, 8.25]$, and satisfying the capacity constraint, produce the same payoff for S. Any strategy placing a positive mass on 0 results in a performance tie at 0. An arbitrarily small increase in performance breaks the tie and increases S's probability of winning. Thus, placing mass on 0 is not optimal for S. Any performance level in excess of 8.25 produces the same probability of winning the rank contest, 1, as a performance of 8.25, and uses more of S's capacity. A simple calculation shows that S's capacity constraint is also satisfied.

Now consider the marginal effect of increased capacity on the payoffs of the two managers. When $X_W = 0$ (which occurs with probability $1 - {}^{\mu_W}/_{\mu_S}$), S wins the rank reward with probability 1, and when W randomizes uniformly over $[0, 8.25]$ (which occurs with probability ${}^{\mu_W}/_{\mu_S}$), S and W win the rank reward with equal probability. Thus, the payoffs to S and W are given as follows:

$$\text{Payoff}_S^{\text{rnk}} = \left(1 - \frac{\mu_W}{\mu_S}\right) + \frac{\mu_W}{\mu_S}\frac{1}{2} = 1 - \frac{1}{2}\frac{\mu_W}{\mu_S}, \quad \text{Payoff}_W^{\text{rnk}} = \frac{1}{2}\frac{\mu_W}{\mu_S}.$$

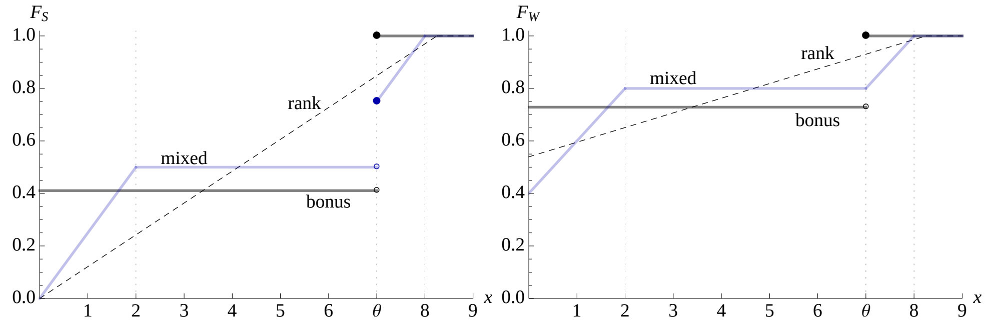
Differentiating these payoff functions shows that

$$\frac{\partial \text{Payoff}_S^{\text{rnk}}}{\partial \mu_S} = \frac{\mu_W}{\mu_S}\frac{\partial \text{Payoff}_W^{\text{rnk}}}{\partial \mu_W} < \frac{\partial \text{Payoff}_W^{\text{rnk}}}{\partial \mu_W}.$$

Hence, in contrast to the bonus competition (Section 2.1.1), in the rank competition, the marginal value of increased capacity is less for S than for W. This result comports well with intuition: letting Usain Bolt (world's top sprinter) start on the 10 meter line in a 100 meter race with a club sprinter, would have little effect on Usain's probability of winning. Letting the club sprinter start at the 10 meter line would have a huge effect on the club sprinter's probability of winning.

2.1.4 Mixed competitions

Do the heuristic observations in this section provide significant insights into the qualitative features of equilibrium risk taking in mixed contests? To answer this question, we consider our model solution, provided by Lemma 1 in Section 3.2, under the assumed parameters of the example (see Table 1) when mixed rewards take the form $B = 1$ and $R = 1$. We illustrate equilibria of the bonus, rank, and mixed contests in Figure 1 and present the equilibrium in the caption of the Figure.



A. Performance Distributions: S *B.* Performance Distributions: W

	Incentives		Ruin Risk		Length(Support)	
	B	R	$\mathbb{P}[X_S = 0]$	$\mathbb{P}[X_W = 0]$	$\lambda[\text{Supp}(F_S)]$	$\lambda[\text{Supp}(F_W)]$
Bonus	1	0	0.41	0.73	0.00	0.00
Mixed	1	1	0.00	0.40	3.00	3.00
Rank	0	1	0.00	0.54	8.25	8.25

C. Statistics

Figure 1: Risk taking in the example. In Panels A and B, equilibrium performance distributions under bonus, rank, and mixed rewards are plotted for W (Panel A) and S (Panel B). Panel C provides summary statistics. Equilibrium performance distributions: with probability 0.20, $X_W = 0$, with probability 0.20, $X_W \sim \text{Uniform}[7, 8]$, with probability 0.40, $X_W \sim \text{Uniform}[0, 2]$; with probability 0.25, $X_S = 7 = \theta$, with probability 0.25, $X_S \sim \text{Uniform}[7, 8]$, with probability 0.50, $X_S \sim \text{Uniform}[0, 2]$.

As shown in Panels A and B of Figure 1, consistent with the observations in Section 2.1.2, the length of the common support for S and W's performance distributions (i.e., their Lebesgue measure) is larger under mixed rewards than under bonus competition. The jump (continuity) in S's (W's) performance distribution at the bonus threshold, $\theta = 7$ observed in Panel A (B) under mixed rewards, reveals that S targets the bonus but W does not. This result is expected given the analysis in Sections 2.1.1 and 2.1.3, which showed that, relative to W, S has a lower marginal gain from devoting capacity to rank competition and the same marginal gain from devoting capacity to capturing bonus rewards.

Although the introduction of rank rewards into the bonus competition lowers ruin risk taking, Panel C shows that ruin risk taking is smaller under mixed rewards than under rank rewards. The presence of bonus rewards in mixed competitions induces S to choose a less spread out, less rank efficient, more bonus efficient, performance distribution. As observed in Section 2.1.3, ruin risk-taking by W is proportional to the rank efficiency of S's performance. Reducing the rank-efficiency of S's risk taking strategy reduces W's ruin risk taking.

2.2 Formalization of the assumptions

We now formalize our analysis. We represent the supports of the performance distribution of S and W by Supp_S and Supp_W respectively; we represent the indicator function for a set of the form $[a, \infty)$, $a \in \mathbb{R}$ by $\mathbb{1}_a$, i.e., $\mathbb{1}_a(x) = 1$ if $x \geq 0$ and equals 0 otherwise.

Remark 1. A very standard result in the all-pay and risk-taking-contest literature is that, in equilibrium, managers never submit performance distributions that result in a positive probability of tied performance. If the managers tie with positive probability, a manager could shift probability mass from the tie point to a higher performance level arbitrarily close to the tie point. This shift would cause the manager's payoff to jump up, while the capacity used to make the shift could be made arbitrarily small. In Appendix Section G we provide a derivation of this standard result in the context of our model.

Per Remark 1, tied performance cannot occur in equilibrium. Thus, the expected rank reward received by S (W) given performance x by the manager S (W), conditioned on the rival manager, W (S) choosing performance distribution F_W (F_S) is given by $R\,\mathbb{P}(x > X_W) = R\,\mathbb{P}(x \geq X_W) = R\,F_W(x)$ ($R\,\mathbb{P}(x > X_S) = R\,\mathbb{P}(x \geq X_S) = R\,F_S(x)$). The bonus payment B is captured if and only if performance weakly exceeds the bonus threshold, θ. Thus, the manager's bonus reward given performance x is given by $B\,\mathbb{1}_\theta(x)$.

We denote the reward from submitting a performance level of x to the strong and weak manager by $\Pi_S(x)$ and $\Pi_W(x)$ respectively, and term Π_S and Π_W the managers' *reward functions*. Combining bonus and absolute performance rewards specified above, shows that these reward functions are given as follows:

$$\Pi_S(x) \coloneqq R\,F_W(x) + B\,\mathbb{1}_\theta(x), \quad \Pi_W(x) \coloneqq R\,F_S(x) + B\,\mathbb{1}_\theta(x). \tag{2.1}$$

Because managers are risk neutral, their payoffs are given by the expectation of the rewards produced by the performance distributions. Thus, the payoffs to the two managers are given by

$$\int_{\mathbb{R}^+} \Pi_i(x)\,\mathrm{d}F_i(x), \qquad i = S, W. \tag{2.2}$$

A manager's problem can be formulated as choosing a performance distribution over the non-negative real line which maximizes her payoff, subject to a capacity constraint that bounds her expected performance. Let $\mathcal{F}^+$ represent the set of distributions with non-negative support, i.e.,

$$\mathcal{F}^+ = \{F : \; F(0-) = 0\}.$$

Consequently, the feasible set of distributions for manager $i = S, W$ is given by $\mathcal{F}_i$, where

$$\mathcal{F}_i = \{F \in \mathcal{F}^+ : \int_{\mathbb{R}^+} x\,\mathrm{d}F(x) \leq \mu_i\}, \quad i = S, W. \tag{2.3}$$

The contest reward functions, defined by equation (2.1), are homogeneous in rewards. Thus, equilibrium behavior of the contestants is determined by the rank-to-bonus ratio, ${}^{R}\!/_{B}$. For this reason, henceforth, to eliminate unnecessary notation, in mixed competitions, we will use the bonus rewards as numéraire, and represent the relative strength of rank and bonus incentives by the *rank focus parameter*, $r = {}^{R}\!/_{B}$. Slightly abusing notation, we also use $r = \infty$ to represent rank competitions and use $r = 0$ to represent bonus competitions.

Because tied performance cannot occur in equilibrium (Remark 1), in any equilibrium, the probability that manager $i = S, W$ wins the rank competition when i's performance equals x is determined by the distribution function of the rival manager j's performance distribution, $F_j(x) := \mathbb{P}[X_j \le x]$. If manager i wins the rank competition, manager i receives r, the rank focus parameter. Because the bonus reward is the numéraire, if the manager captures the bonus, the manager receives 1. Manager i captures the bonus if and only if $x \ge \theta$, i.e., $\mathbb{1}_\theta(x) = 1$.

Thus, a pair of performance distributions, (F_S^*, F_W^*), is an *equilibrium* if the probability of tied performance under (F_S^*, F_W^*) is zero and each performance distribution is a best reply to the other, i.e., for $i = S, W$,

$$\int_{\mathbb{R}^+} \Pi_i(x)\, \mathrm{d}F_i^*(x) = \operatorname*{Max}_{F \in \mathcal{F}^+} \int_{\mathbb{R}^+} \Pi_i^*(x)\, dF(x) \tag{2.4}$$

$$\text{s.t.} \int_{\mathbb{R}^+} x\, \mathrm{d}F(x) \le \mu_i, \quad \text{where} \tag{2.5}$$

$$\Pi_S^*(x) = r\, F_W^*(x) + \mathbb{1}_\theta(x),\ \Pi_W^*(x) = r\, F_S^*(x) + \mathbb{1}_\theta(x). \tag{2.6}$$

Our aim is to characterize equilibrium manager performance distributions. Our characterization builds on a large literature on rank competitions and somewhat smaller but still significant literature on risk-taking contests. In order to avoid exhausting readers' patience, we will start by stating, in Remark 2, a few fairly obvious, known restrictions that these frameworks place on equilibrium performance distributions. In Appendix Section G, we provide formal derivations of these restrictions.

Remark 2. (a) With the possible exception of performance $x = \theta$, the supports of the managers' performance distributions will coincide. (b) $\mathrm{Supp}_i \cap [0, \theta)$ and $\mathrm{Supp}_i \cap [\theta, \infty)$, $i = S, W$, are connected. (c) manager performance distributions are continuous except perhaps at 0 and θ. (d) 0 must be in the support of both managers' performance distributions.

The logic behind these restrictions is that, when small changes in positive performance levels do not affect the probability of receiving the bonus reward, managers' incentives are determined by the following characteristics of rank competitions: first, in rank competitions, topping rivals' performance by a small amount garners the same reward as topping by a large amount, and topping by a large amount uses more capacity. Thus, when competing for rank dominance, managers "shadow" each other's performance. Second, an increase in performance not accompanied by an increase in the probability of topping rivals is never optimal. So over regions where the rank reward does not vary, the supports of the manager's performance distributions are connected. Third, because discontinuities in rivals' performance strategies imply an infinite marginal gain from arbitrarily small increases in a manager's performance, such discontinuities cannot occur in equilibrium.

Fourth, these observations imply that 0 is in the support of both managers' performance distributions. If 0 were not in the support of one of the manager's performance distributions, say manager S, then, because Supp_S is closed by definition, it would be the case that $a = \sup_x\{F_S(x) = 0\} > 0$. Because neither manager has sufficient capacity to submit performance in weakly excess of θ, the bonus target, with probability 1, $a < \theta$. So a would be in the support of S's performance distribution and $a \in (0, \theta)$. Performance distributions are continuous over $(0, \theta)$ (Remark 2.c), thus $F_S(a) = F_S(0) = 0$. Inspecting W's reward function, (2.6), shows that the payoff to W from performance equal to a would be the same as the payoff to W from submitting performance equal to 0 and, because $a > 0$, submitting performance a would use up more capacity. Thus, a could not be a best response for W and thus would not be in Supp_W. However, by Remark 2.a, a being in Supp_S but not in Supp_W is impossible.

Remark 3. The basic tool for identifying equilibrium distributions is a multiplier characterization of best responses: there exist *multipliers*, $\alpha_i \geq 0$ and $\beta_i > 0$, satisfying the following conditions:

(i) For all $x \geq 0$, $\Pi_i(x) \leq \alpha_i + \beta_i\, x$.
(ii) If x is in the support of manager i's equilibrium performance distribution, then $\Pi_i(x) = \alpha_i + \beta_i\, x$.

We call the map $x \hookrightarrow \alpha_i + \beta_i\, x$ manager i's *support line*, and represent the support lines with ℓ_i. These observations imply (a) each manager's support line majorizes the manager's reward function, and (b) if x' and x'' are two performance levels in the support of i's performance distribution, $\Pi_i(x'') - \Pi_i(x') = \beta_i\,(x'' - x')$, i.e., for a given manager, the marginal gain from submitting any two performance levels in the support of the performance distribution is constant and equals to β_i. Since submitting a constant performance distribution equal to capacity is always feasible, the expected payoff to a manager equals $\alpha_i + \beta_i\, \mu_i$. This implies that the marginal gain from increased capacity equals β_i, the slope of manager i's support line. In Appendix Section G, we provide a formal justification for this approach.

2.3 Properties of equilibrium performance distributions

The results developed thus far, particularly the support-line characterization of equilibrium performance distributions, impose strong restrictions on performance distributions in all of the equilibrium configurations we analyze in the baseline model setting. The proof of this result, and all subsequent results, is presented in the appendix to this paper.

Proposition 1 *Let (F_S, F_W) be a pair of equilibrium performance distributions. (i) For almost all $x \geq 0$, $F'_W(x) = 0$ or $F'_W(x) = \beta_S/r$; $F'_S(x) = 0$ or $F'_S(x) = \beta_W/r$. (ii) The equilibrium distribution functions for managers $i = S, W$ can always be expressed as follows:*

$$F_i(x) = \gamma_0 \, \mathbb{1}_0(x) + \gamma_\theta \, \mathbb{1}_\theta(x) + \int_0^x F'_i(t) \, dt, \text{ where } \gamma_0 \geq 0, \; \gamma_\theta \geq 0, \text{ and } \gamma_0 + \gamma_\theta \leq 1.$$

(iii) The capacity constraint, equation (2.5)*, is binding.*

Because the support of a manager's performance distribution contains at most two connected components (Remark 2.b), Proposition 1 implies that, over any interval in the supports of the managers' performance distributions, the density of performance is (almost everywhere) constant. Thus, the continuous component of the managers' performance distributions can be represented by mixtures of uniform distributions. Uniformity results from two properties of the baseline model: (a) performance is constrained by an upper bound on a linear function of performance, its expectation, and (b) the rank reward of each manager is a multiple of the performance distribution of a single rival manager. Because of property (a), the support lines are affine in performance. Because managerial performance is concentrated on performance levels where the reward function meets the manager's support line, the continuous component of the reward function must be linear on the support of each manager's performance distribution. Given property (a), property (b) implies that the continuous component of the rival manager's performance distribution must be linear and thus can be represented as a mixture of uniform distributions.

When we extend our analysis (Section 5.1) to allow for nonlinear constraints on performance, the extended model does not have property (a); when we extend our analysis (Section 5.4.2) to encompass competitions among many managers, the extended model does not have, property (b). In these extensions, the continuous component of managerial performance distribution is generally not linear. However, the characterizations of managerial risk taking developed the baseline setting are robust to the introduction on non-linear performance constraints and many competitors, as is the fairly obvious result recorded in part (iii), i.e., in these settings, the capacity constraint also binds.

3 Configurations

3.1 Benchmark: Pure bonus competition

Recall that, in the benchmark bonus competition case discussed in Section 2.1.1, both managers use all their capacity to target the bonus. This rather obvious result is recorded below in order to establish the benchmark for measuring the effect of introducing rank rewards on the behavior of managers receiving bonus rewards.

Result 1. In bonus competitions, i.e., when $r = 0$, manager $i = S, W$ places probability weight μ_i/θ on bonus target, θ, probability weight $1 - \mu_i/\theta$ on 0.

3.2 Configuration Eq2: Both S and W chase the bonus

When sufficiently small rank rewards are introduced into a bonus competition, bonus chasing is still attractive to both managers. We call equilibria having this configuration *Eq2 equilibria*. In the Eq2 equilibria, equilibrium performance distributions must satisfy the following conditions: (i) Each manager's reward at the bonus threshold must meet her support line. (ii) The slope of each manager's support line above and below the bonus threshold must be the same. (iii) For both managers, the capacity constraint is binding. (iv) For both managers, all points in the support of their performance distributions lie on their support lines. These conditions yield the following characterization of Eq2 equilibria.

Lemma 1 (**Eq2:** S **and** W **chase the bonus**) *In any equilibrium in which both S and W chase the bonus, the equilibrium performance distributions for S and W satisfy*

$$F_S^* = p_S^h \,\mathrm{Unif}[\theta, u_H] + p_S^\theta \,\mathbb{1}_\theta + (1 - p_S^h - p_S^\theta)\,\mathrm{Unif}[0, u_L],$$
$$F_W^* = p_W^h \,\mathrm{Unif}[\theta, u_H] + p_W^0 \,\mathbb{1}_0 + (1 - p_W^h - p_W^0)\,\mathrm{Unif}[0, u_L],$$

$u_L \in (0, \theta)$, $u_H > \theta$,

$$p_S^\theta = \frac{\mu_S - \mu_W}{\theta + r\,\mu_W} \in (0,1), \qquad p_S^h = \frac{(r+1)\,(u_H - \theta)}{r\,u_H} \in (0,1),$$
$$p_W^0 = \frac{(r+1)\,(\mu_S - \mu_W)}{\theta + r\,\mu_S} \in (0,1), \qquad p_W^h = \frac{u_H - \theta}{r\,(\theta - u_L)} \in (0,1),$$

and u_H *and* u_L *are given as follows:*

$$u_H = \frac{2\,(1+r)\,(\theta + r\,\mu_S)\,(\theta + r\,\mu_W)^2}{(\theta + r\,\mu_S)^2 + (1+r)^2\,(\theta + r\,\mu_W)^2}, \quad u_L = \theta - \frac{2\,(\theta + r\,\mu_S)^2\,(\theta + r\,\mu_W)}{(\theta + r\,\mu_S)^2 + (1+r)^2\,(\theta + r\,\mu_W)^2}. \tag{3.1}$$

In Eq2 configurations, managers compete for rank dominance both above and below the bonus threshold, θ. Over the *subthreshold competition region*, $[0, u_L]$, the weak manager places positive weight on zero performance, the strong manager does not and, conditioned on submitting performance in $(0, u_L]$, both managers choose the same distribution of performance. However, their probabilities of submitting performance in $(0, u_L]$ differ: the strong manager's probability equals $1 - p_S^h - p_S^\theta$ and the weak manager's probability equals $1 - p_W^h - p_W^0$.

Over the *superthreshold region*, $[\theta, u_H]$, the strong manager *targets the bonus* by placing positive probability weight p_S^θ exactly on the bonus threshold and the weak manager does not target the bonus. Both managers choose the same performance distribution conditioned on performance in $(\theta, u_H]$, but target this region with different probabilities, p_S^h for the strong manager and p_W^h for the weak manager.

The distributions specified in Lemma 3 are best replies if and only if the graphs of the managers' reward functions lie weakly below their support lines and, at all performance levels, x, in the support of their performance distributions, the support line evaluated at x equals the reward function evaluated at x.

These optimality conditions are illustrated in Figure 2. Because the rank-based reward is earned if and only if the performance tops rival's performance and ties do not occur in equilibrium, the probability that a manager wins the rank reward for a performance level of x is $F_j(x)$ where j represents the performance distribution of the rival. The reward functions of the managers jump up by 1 at $x = \theta$, the point where the bonus threshold is attained.

The vertical-axis intercept of W's support line, $\alpha_W = 0$, and the vertical axis intercept for S's support line, $\alpha_S > 0$, imply that $F_S(0) = 0$ and $F_W(0) > 0$, i.e., consistent with Lemma 1, W accepts ruin risk and S does not. In Figure 2 the slope of W's support line, β_W, exceeds the slope of S's support line, β_S. Thus, per Remark 3 and consistent with the intuition developed in Section 2.1, W's marginal gain from increased capacity is greater than S's.

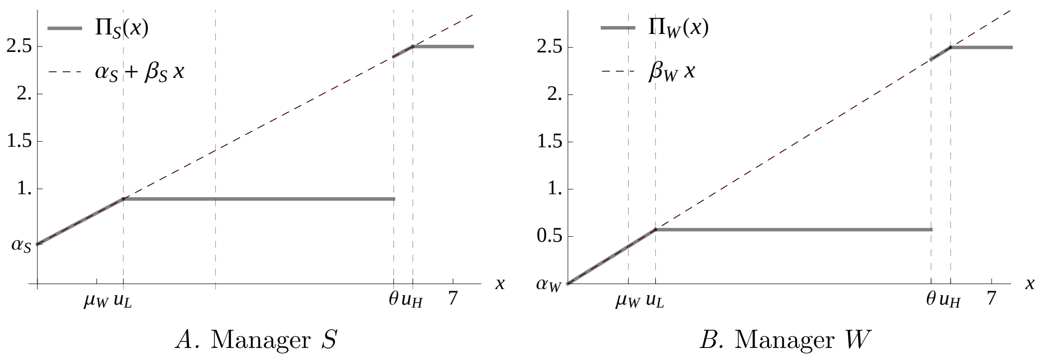


Figure 2: Equilibrium rewards in Eq2 equilibria, where both S and W chase the bonus. The figure illustrates the reward functions, Π_i and support lines, $\alpha_i + \beta_i x$ for manager $i = S$ (Panel A) and manager $i = W$ (Panel B). The horizontal axis represents performance, x. The parameters are $\mu_S = 3$, $\mu_W = 1$, $\theta = 6$, $r = 2/3$.

3.3 Configuration Eq1: Only S chases the bonus

As the rank focus parameter, r, or the bonus threshold, θ, increases, bonus chasing may cease to be attractive to one of the managers. We call equilibria with this property *Eq1 configurations.* Our first result on the Eq1 configuration is that, when only one manager chases the bonus, the bonus chasing manager must be the strong manager.

Lemma 2 *There do not exist equilibria in which only the weak manager chases the bonus.*

The logic for this result is simple: the strong manager's marginal gain from applying capacity to rank competition is lower than the weak manager's. Thus, if the weak manager is willing to chase the bonus, so is the strong manager. Our next result, Lemma 3, specifies equilibrium strategies when only the strong manager, S, chases the bonus.

Lemma 3 (Eq1: S but not W chases the bonus) *In any equilibrium in which only S chases the bonus, the equilibrium performance distributions for S and W satisfy*

$$F_S^* = p_S^\theta \, \mathbb{1}_\theta + (1 - p_S^\theta)\, \mathrm{Unif}[0, u], \;\; F_W^* = p_W^0 \, \mathbb{1}_0 + (1 - p_W^0)\, \mathrm{Unif}[0, u], \;\; \textit{where}$$

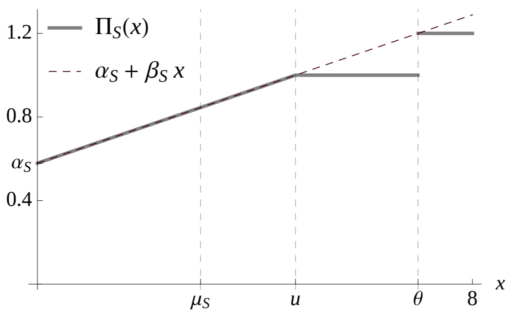
$$u = \frac{2\,\theta\,\mu_W}{\mu_W + \sqrt{2\,\theta\,\mu_W/r + \mu_W^2}} < \theta, \qquad p_S^\theta = \frac{2\,\mu_S - u}{2\,\theta - u} \in (0,1), \qquad p_W^0 = 1 - \frac{2}{u}\,\mu_W \in (0,1).$$

As in the Eq2 configuration, the weak manager places positive weight p_W^0 on zero performance; the strong manager places no weight on zero performance. The strong manager targets the bonus with probability p_S^θ; the weak manager never captures the bonus. Conditioned on performance in $(0, u]$, $u < \theta$, both managers choose the same distribution of performance. However, their probabilities of targeting $(0, u]$ differ: the strong manager targets $(0, u]$ with probability $1 - p_S^\theta$ and the weak manager with probability $1 - p_W^0$.

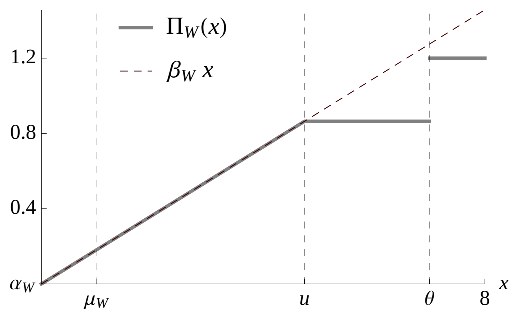
Because part of the strong manager's capacity is diverted to targeting the bonus, the upper bound on performance below the bonus threshold, u, must be less than $2\,\mu_S$, the upper bound of performance under rank competition. Contest reward functions for this equilibrium configuration are illustrated by Figure 3.

3.4 Configuration Eq0: Neither S nor W chases the bonus

If the rank focus parameter or the bonus threshold is sufficiently high, neither manager will chase the bonus, resulting in an Eq0 configuration. In this configuration, the contest reduces to a purely rank-based contest. Both managers play the rank competition strategies discussed in Section 2.1.3, i.e., the same strategies that they would play in the absence of bonus compensation. Lemma 4 characterizes equilibrium strategies in this configuration.



A. Manager S



B. Manager W

Figure 3: Equilibrium rewards in Eq1 equilibria, where only S chases the bonus. The figure illustrates the reward functions, Π_i and support lines, $\alpha_i + \beta_i\, x$ for manager $i = S$ (Panel A) and manager $i = W$ (Panel B). The horizontal axis represents performance, x. The parameters are $\mu_S = 3$, $\mu_W = 1$, $\theta = 7$, $r = 5$.

Lemma 4 (**Eq0: Neither manager chases the bonus**) *In any equilibrium in which neither S nor W chase the bonus, the equilibrium performance distributions for S and W satisfy*

$$F_S^* = \mathrm{Unif}[0, 2\,\mu_S],$$
$$F_W^* = \left(1 - \frac{\mu_W}{\mu_S}\right)\mathbb{1}_0 + \frac{\mu_W}{\mu_S}\,\mathrm{Unif}[0, 2\,\mu_S].$$

In this equilibrium configuration, the strong manager randomizes uniformly over $[0, 2\,\mu_S]$. The weak manager does not have sufficient capacity to emulate this performance distribution. The weak manager mimics the strong manager's performance distribution to the extent his capacity permits. This requires "paying" for the high performance associated with being a strong manager by putting probability mass $(\mu_S - \mu_W)/\mu_S$ on 0, i.e., accepting ruin risk.

3.5 Existence and uniqueness of configurations

Our next result shows that, for any given set of parameters, one, and only one, of the three configurations, Eq0, Eq1, and Eq2, characterizes equilibrium behavior.

Lemma 5 *Any choice of admissible bonus packages, i.e., (θ, r), such that $\theta > \mu_S$ and $r > 0$, sustains one and only one equilibrium configuration, Eq2, Eq1, or Eq0.*

In Appendix Section A we show that, although the regions supporting the three configurations are defined by implicit polynomial equations of fairly high order, fairly simple characterizations can be obtained using parametric curves in θ–r space that map out the boundaries between the regions.

In Figure 4, we provide graphs of these curves. The detailed analytical form of these boundary curves is deferred to Appendix Section A. In Panel A, the ratio between the strength of the strong and weak manager is 2:1; in Panel B, the ratio is 7:1.

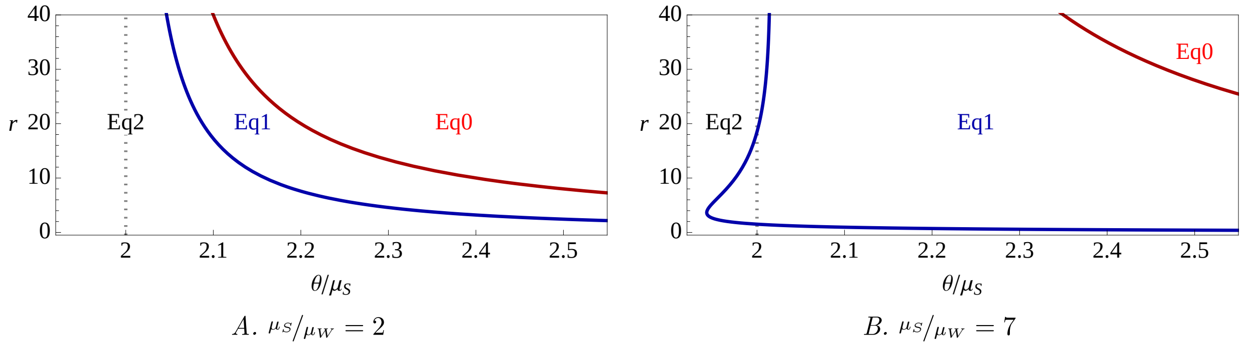


Figure 4: Parametric curves for equilibrium configurations. In the figure, the blue line represents the Eq1/Eq2 boundary and the red line represents the Eq0/Eq1 boundary. In Panel A, $(\mu_S, \mu_W) = (2, 1)$. In Panel B, $(\mu_S, \mu_W) = (7, 1)$. The dotted vertical line represents points where $\theta = 2\,\mu_S$, maximum performance under rank competition.

In Panel A, where strength asymmetry is not very extreme, the curve representing the Eq1/Eq2 boundary never crosses the dashed line at $\theta = 2\,\mu_S$, the rank-competition upper bound. In Panel B, where strength asymmetry is very extreme, the curve bends back and transversally intersects the $\theta = 2\,\mu_S$ line. Thus, in Panel A, Eq1 configurations can be sustained only when the bonus threshold, θ exceeds $2\,\mu_S$. However, in Panel B, for some choices of the rank focus parameter, r, Eq1 equilibria can be sustained even at bonus thresholds, θ, less than the rank competition upper bound, $2\,\mu_S$.

4 Risk taking in mixed competitions

4.1 The effects of mixed rewards on ruin risk taking

We initiate our analysis of risk-taking by considering ruin risk, the probability that managers assign to zero performance. The motivation for our focus on ruin risk is simply that this is the sort of risk taking that is the primary concern of financial economists, regulators, and practitioners. Much of the research on risk taking has been motivated by concerns about the consequences of, for example, fund managers taking extreme tail risk to boost alpha, CEOs increasing bankruptcy risk by gambling on resurrection. However, the effects of mixed rewards on overall risk, measured by stochastic dominance, are also important and will be the subject of Section 4.3.

Proposition 2 summarizes the mollifying effects of mixed rewards on ruin risk taking. Mathematically, the proposition follows almost directly from the characterizations of equilibrium performance strategies provided by Lemmas 1, 3, and 4.

Proposition 2 (Ruin risk taking) *In mixed competitions,* *(i) If the equilibrium configuration is Eq0, (i.e., neither manager chases the bonus), the probability of ruin risk taking, $1 - \mu_W/\mu_S$, is less than the probability of ruin risk taking under bonus competition and equal to the probability under rank competition.*

(ii) If the equilibrium configuration is either Eq1 or Eq2 (i.e., at least one manager chases the bonus), the probability of ruin risk taking, p_W^0, is less than the probability of ruin risk taking in both rank and bonus competitions.

The driver for the lower level of ruin risk in mixed competitions relative to bonus competitions is fairly straightforward: when rank rewards are introduced into bonus competitions, bonus targeting strategies funded by accepting significant ruin risk have significant opportunity costs in terms of lost rank rewards. These opportunity costs reduce ruin risk taking and lead strong managers to eschew ruin risk taking entirely.

The driver for the lower level of ruin risk in mixed competitions relative to rank competitions is more subtle: increased rank focus, r, leads the strong manager to adopt more rank efficient performance strategies that place less weight on bonus targeting. The increased rank efficiency of the strong manager's performance forces the weak manage to accept more ruin risk to effectively compete. We illustrate this effect in Figure 5.

In the figure, we hold the bonus threshold, θ, fixed. For each level of rank focus parameter, r, that supports an Eq1 or Eq2 equilibria, we plot the ordered pair $(p_S^\theta(r), p_W^0(r))$, representing the levels of bonus targeting, p_S^θ, and ruin risk taking, p_W^0, corresponding to r. The figure illustrates the inverse relationship between bonus targeting by the strong manager and ruin risk taking by the weak manager. As rank rewards increase, the strong manager devotes less and less capacity to bonus chasing, a very inefficient rank-competition strategy, and instead applies capacity to rank domination. This effect forces the weak manager accept more ruin risk to "keep up."

4.1.1 How low can ruin risk taking go?

As our example in Section 2.1 illustrates, ruin risk taking under mixed rewards can be substantially less than ruin risk taking in rank or bonus competitions. How low can ruin risk taking go? In other words, what is the lower bound on ruin risk taking over all admissible choices of the rank focus, r, and bonus threshold, θ, parameters? To answer this question, we need to consider the effect of the bonus threshold, θ, on ruin risk taking.

Lemma 6 *(i) When the equilibrium configuration is Eq1, increasing the bonus threshold, θ, increases ruin risk taking, p_W^0. (ii) When the equilibrium configuration is Eq2, increasing the bonus threshold, θ, decreases ruin risk taking, p_W^0.*

The intuition for this result is the intuition developed in Section 2.1. In the Eq1 configuration, only S chases the bonus. Thus, a marginal increase in θ, which makes bonus chasing less

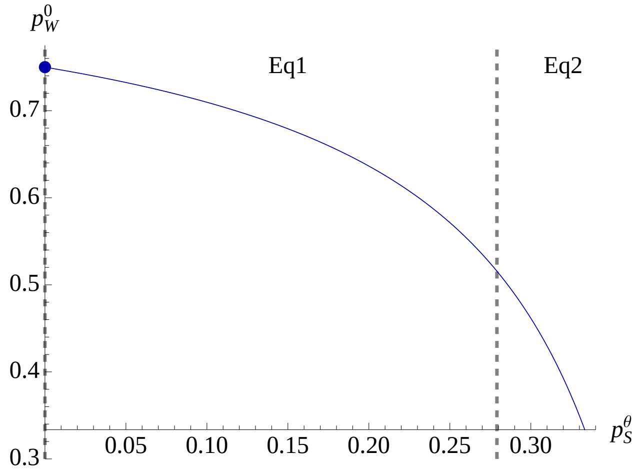


Figure 5: Bonus targeting by S and ruin risk taking by W. The figure plots a parametric curve where the parameter is the rank focus of the managers, r. The abscissa of each point on the curve represents the probability the strong manager targets the bonus, p_S^θ, and the ordinate represents the ruin risk accepted by the weak manager, p_W^0. The dashed line divides the plane between the points on the curve under which the level of rank focus supports an Eq1 or Eq2 configuration. The point at the left end of the curve represents bonus targeting and ruin risk taking in an Eq0 configuration. The fixed parameters in the figure are $\mu_S = 4$, $\mu_W = 1$, and $\theta = 9$.

attractive to both managers, has no effect on the capacity W devotes to bonus chasing. The increase in θ however makes diverting capacity to bonus chasing less attractive for S, leading S to focus more on rank competition, and this increases ruin risk taking by W.

When the equilibrium configuration is Eq2, the balance of incentives changes. In the Eq2 configuration both S and W chase the bonus. Increasing θ makes bonus competition less attractive to both S and W. However, as discussed in Section 2.1, the marginal gain from applying capacity to rank competition, relative to chasing the bonus, is smaller for S than for W. Thus, W responds to the increased threshold by reducing capacity devoted to bonus chasing more than S. The capacity transferred to rank competition by W reduces the ruin risk W must accept to effectively compete for rank dominance and thus reduces ruin risk taking.

Consequently, if equilibrium configuration is Eq1, reducing the bonus threshold lowers ruin risk taking. When the configuration switches to Eq2, further decreases in the threshold increase ruin risk taking. Thus, it is natural to conjecture that choices of r and θ on the boundary between these two configurations minimize ruin risk taking. In fact, as the next proposition shows, by choosing combinations of the parameter, r, and the bonus threshold, θ, that support equilibria on the Eq1/Eq2 boundary, it is possible to make ruin risk taking arbitrary small.

Proposition 3 *(i) Ruin risk taking is present in all equilibria (i.e., $p_W^0 > 0$). (ii) However, there exists, a sequence $\{(\theta_n, r_n)\}$ under which ruin risk taking approaches 0, i.e., as $n \to \infty$,*

$p_W^0(n) \to 0.$

Absent rank rewards, high-powered bonus rewards engender significant ruin risk taking. Absent bonus rewards, rank rewards generate significant ruin risk taking. Thus Proposition 3 shows that mixtures of very high powered bonus rewards and large rank dominance rewards can lead to negligible ruin risk taking.

4.2 Upside risk taking

In contrast to ruin risk taking, upside risk taking, measured by the maximum performance level in the supports of the managers' performance distributions, is generally not reduced by the introduction of rank rewards. The effect of rank rewards on upside risk taking depends on the degree of strength asymmetry between the managers, measured by ${}^{\mu_S}/_{\mu_W}$ and the level of bonus threshold relative to the capacity of the strong manager, S.

Proposition 4 *Consider a mixed contest where the strengths of the managers, μ_S and μ_W, and the bonus threshold, θ, are fixed, and the rank focus parameter, r, varies.*

(i) The upper bound on the performance of the strong manager under mixed rewards is never less than the minimum of its bounds under rank and bonus competition, $\min[2\,\mu_S, \theta]$.

(ii) When strength asymmetry is moderate i.e., $1 < \frac{\mu_S}{\mu_W} \leq 3 + 2\sqrt{2} \approx 5.83$, and the bonus threshold is moderately challenging, i.e., $\mu_S < \theta < 2\,\mu_S$,

(a) The equilibrium configuration is Eq2.

(b) The upper bound on performance of the managers is given by u_H as defined by equation (3.1) *in Lemma 1.*

(c) $u_H(r)$ is initially increasing in r, i.e., $u_H(r) = \theta + \mu_W\, r + o(r)$, $r \to 0$, and ultimately decreasing, i.e., $u_H(r) = 2\,\mu_S + 2\,(\theta - \mu_S)/r + o(r)$, $r \to \infty$.

(d) If, $\mu_S < 3\,\mu_W$ then there exists $r^o > 0$ such that for all $r < r^o$, the upper bound on performance in mixed competitions is less than the upper bound under rank competitions but more than the upper bound under bonus competitions, and for all $r > r^o$, the upper bound on performance in mixed competitions exceeds the upper bound on performance in both rank and bonus competitions.

(iii) When strength asymmetry is extreme, i.e., $\frac{\mu_S}{\mu_W} > 3+2\sqrt{2}$, there exist mixed competitions in which the upper bound on the performance of the weak manager is less than its upper bound under both rank and bonus competition.

Part (i) of Proposition 4 illustrates the contrast between the effects of mixed rewards on ruin risk taking and upside risk taking. As shown in Proposition 2, mixed rewards, quite generally, reduce ruin risk taking relative to both bonus and rank rewards. In contrast, Part (i) of Proposition 4 shows that S's maximum performance always weakly exceeds either the rank or bonus competition maximum performance.

In fact, when reward thresholds and strength asymmetry are modest, Proposition 4.ii shows that mixed rewards lead to maximum performance that exceeds the bonus competition maximum and, when rank rewards are sufficiently large, maximum mixed competition performance exceeds both the rank and bonus maxima. The logic behind this result is fairly transparent: when the bonus threshold is not excessively high and strength asymmetry is not too extreme, both the strong and weak managers pursue bonus rewards. The presence of rank rewards causes performance to spread out, as discussed in Section 2.1. Because of the reward for topping the bonus threshold provided by the bonus payment, performance spreads out above the bonus threshold. Thus, when rank incentives are weak, i.e., r is close to zero, increasing rank incentives increases maximum performance and eventually maximum performance surpasses the rank competition bound, $2\,\mu_S$. However, ultimately, as the rank focus parameter increases without bound, the subsidy provided by the bonus reward for competing for rank dominance above the bonus threshold has a negligible effect on the welfare of the managers, and maximum performance in the mixed competition decreases and converges to the rank competition maximum, $2\,\mu_S$. These effects are illustrated in Figure 6.

The exception to the generally positive effect of mixed incentives on maximum performance is noted in part (iii) of the proposition. When strength asymmetry is extreme, for intermediate levels of the rank focus parameter, r, Eq1 equilibria can exist even when bonus thresholds are moderate, i.e., $\theta \leq 2\mu_S$. By definition, in an Eq1 equilibria, W's maximum performance is less than W's maximum performance in a bonus competition. Because, in Eq1 equilibria, the maximum performance of W is less than W's maximum performance in a rank competition, $2\mu_S$, W's maximum performance in the mixed competition is less than both W's rank and bonus competition maxima. S's maximum performance is equal to S's bonus maximum and less than S's rank maximum.

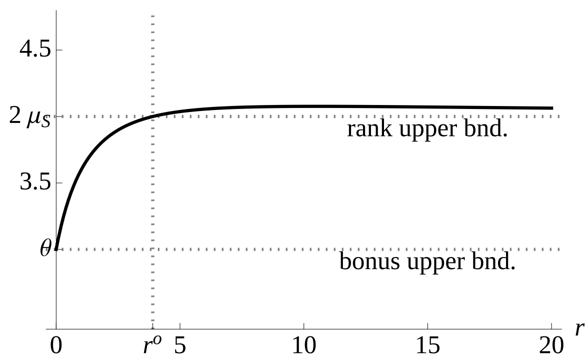


Figure 6: Upside risk taking. In this example, $\mu_S = 3$, $\mu_W = 2$, and $\theta = 3$. $2\,\mu_S$ is the maximum performance level under rank competition; θ represents the maximum performance level under bonus competition. The function graphed represents maximum performance in mixed competitions as a function of the rank focus parameter, r.

4.3 Effect of rank rewards on the overall riskiness of performance

Thus far, our analysis has focused on how mixed rewards affect ruin risk-taking and upside risk taking. In this section, we consider how mixed rewards affect the stochastic ordering of managers' performance distributions. Holding constant the bonus threshold, θ, and the managers' capacities, μ_S and μ_W, we consider, for each manager type, S and W, whether performance under mixed rewards stochastically dominates performance under bonus rewards. Because the managers' capacity do not vary when rewards are varied, and capacity determines expected performance, "size orders" such as first-order stochastic dominance, are not appropriate for these comparisons. Consequently, we focus on higher order stochastic-dominance relations that measure the riskiness of performance: second-order and third-order stochastic dominance.[10]

Our next proposition shows that performance under bonus rewards never, second or third order, stochastically dominates performance under mixed rewards. Under the conditions specified in the proposition, performance under mixed rewards second or third-order stochastically dominates performance under bonus rewards.

Proposition 5 *Let $X_i \overset{d}{\sim} F_i$, $i = S, W$, represent equilibrium performance distributions under mixed rewards and let $Y_i \overset{d}{\sim} G_i$, $i = S, W$, represent equilibrium performance distributions under bonus rewards, (i.e., when $r = 0$), given bonus threshold θ and manager capacities, μ_S and μ_W.*

(i) *X_i and Y_i, $i = S, W$, are never ordered by first-order stochastic dominance.*

(ii) *If the equilibrium configuration is Eq1, then X_i strictly second-order stochastically dominates (SSD) Y_i, $i = S, W$.*

(iii) *If the equilibrium configuration is Eq2,*

- *(a)* $\operatorname{sgn}[\operatorname{Var}[Y_S] - \operatorname{Var}[X_S]] = \operatorname{sgn}[\operatorname{Var}[Y_W] - \operatorname{Var}[X_W]]$.
- *(b)* *X_i and Y_i, $i = S, W$, are never ordered by second-order stochastic dominance.*
- *(c)* *If* $\operatorname{Var}[Y_W] - \operatorname{Var}[X_W] \geq 0$ *(* $\operatorname{Var}[Y_W] - \operatorname{Var}[X_W] > 0$*) , then X_i third-order (strictly) stochastically dominates (TSD) Y_i, $i = S, W$.*
- *(d)* *If* $\operatorname{Var}[Y_W] - \operatorname{Var}[X_W] < 0$*, X_i and Y_i, $i = S, W$, are not ordered by third-order stochastic dominance.*

Proposition 5 shows that, when mixed rewards result in an Eq1 configuration, performance under mixed rewards is less risky, in the sense of Rothschild and Stiglitz (1970), than performance under bonus rewards. In Eq1 configurations, W eschews chasing the bonus entirely and focuses on rank competition; S diverts some capacity from bonus targeting to competing

[10] If X and Y are random variables and v is a utility-of-wealth function, X second-order stochastically dominates (SSD) Y if $\mathbb{E}[v(X)] \geq \mathbb{E}[v(Y)]$ whenever $v' > 0$ and $v'' < 0$. X third-order stochastically dominates (TSD) Y if $\mathbb{E}[v(X)] \geq \mathbb{E}[v(Y)]$ whenever $v' > 0$, $v'' < 0$, and $v''' > 0$, e.g., v represents a CARA or CPRA utility-of-wealth function.

with W for rank rewards at performance levels less than the bonus threshold. These effects reduce the riskiness of both managers' performance.

In Eq2 configurations, the situation is a bit more complex. Ruin risk is less under mixed rewards (Proposition 2). Hence, bonus rewards can never second-order stochastically dominate mixed rewards. However, the upper bound on performance under mixed rewards is also higher than the upper bound under bonus rewards, i.e. mixed reward contests generate upper tail risk. Thus, performance under mixed and bonus rewards is not ordered by second-order stochastic dominance.

Because, relative to bonus competitions, in Eq2 configurations, ruin risk is lower and the upper bound on performance is higher, mixed competitions third-order stochastically dominate (TSD) bonus competitions whenever variance of performance under mixed rewards is (weakly) smaller.

The algebraic expressions for performance variance under mixed rewards are very non-intuitive ratios of high degree polynomials. Hence, in Figure 7, we simply illustrate the parameter region in which performance under mixed rewards dominates performance under bonus rewards under TSD.

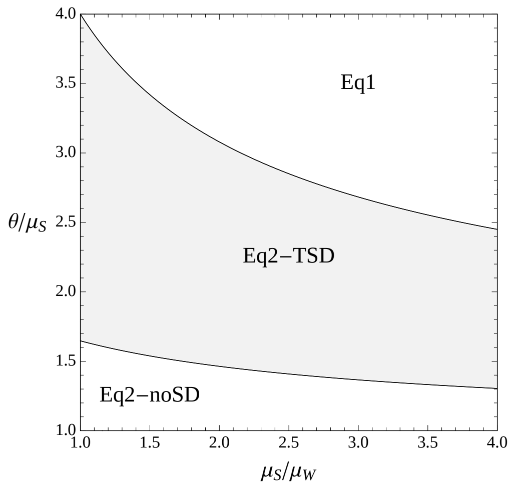


Figure 7: Stochastic dominance and parameter regions. In the figure, the gray shaded region labeled 'Eq2-TSD' represents the region in which the equilibrium configuration under mixed rewards is Eq2 and the performance distributions of both the strong and weak managers third-order stochastically dominate their performance distributions under bonus rewards. The region labeled "Eq1" represents the region where the equilibrium configuration under mixed rewards is Eq1, and the region labeled "Eq-2-noSD" represents the region where the equilibrium configuration under mixed rewards is Eq2 but performance distributions are not ordered by first, second, or third-order stochastic dominance. In the figure, the rank and bonus rewards are equal when rewards are mixed, i.e., $r = 1$.

In the figure, performance under mixed rewards in Eq2 configurations third-order stochastically dominates performance under bonus rewards except when the bonus threshold, θ, is low relative to the capacity of S, and thus risk taking incentives under bonus rewards are moderate, and the strength asymmetry between S and W is small, and thus the risk taking incentives under rank competition are also moderate. This observation is consistent with our earlier results on ruin risk—mixed rewards mollify risk taking most when rank or bonus rewards would produce extreme risk taking. One simple sufficient condition for performance in mixed competitions to third-order stochastically dominate performance in bonus competitions is provided by the following corollary.

Corollary 1 *If strength asymmetry is moderate (see Proposition 4) and* $(4/3)\,\mu_S < \theta < 2\,\mu_S$*, there exists* $r^o > 0$ *such that if* $r > r^o$*, then performance in mixed competitions third-order stochastically dominates performance in bonus competitions.*

5 Robustness of the baseline model

5.1 Non-expectational capacity constraints

In some contexts, e.g., asset substitution/risk-shifting models in corporate finance, risky strategies dissipate value. Our baseline analysis assumes that the only restriction on manager performance is an expectational restriction on performance. Thus, if a performance distribution is feasible, a strategy with the same expected value but higher risk is also feasible. Consequently, the expectational constraint raises a natural question: would constraints that penalized risk taking lead to different conclusions than the baseline model? In this section, we address this question by extending our analysis to encompass non-linear constraints on performance.

In order to define competitions with non-linear constraints on performance, we start by defining a capacity function.

Definition 1. $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a *capacity function* if (a) $\Psi(0) = 0$, (b) Ψ is thrice differentiable and strictly increasing, (c) $\Psi(x) \to \infty$ as $x \to \infty$.

We define a *non-linear constraint contest* as follows: In a non-linear contest, the contest and contest equilibrium are defined as in the baseline model with one exception—expectational capacity constraint (equation (2.3) in the baseline model) is replaced with the non-linear constraint given by equation (5.1) below:

$$\mathbb{E}[\Psi(X_i)] \leq c_i, \quad c_i > 0,\ i = S, W,^{11} \tag{5.1}$$

where Ψ is a capacity function.

Note that, if in a non-linear constraint contest, Ψ is strictly convex, a performance distribution being feasible does not imply that a mean-preserving spread of performance distri-

[11] We use c_i instead of μ_i to represent capacity because, in this non-linear setting capacity does not bound mean performance.

bution is feasible. A convex capacity function will constrain risk taking by ensuring that riskier performance is more costly, i.e., uses up more capacity, than a less risky performance distributions.

In order to show that our central results are robust to non-expectational constraints, we first show that each non-linear constraint contest corresponds to a contest with an expectational constraint, which we will term a Ψ-contest. The Ψ-contest corresponding to a non-linear constraint contest is defined as follows: let

$$Z_i = \Psi(X_i),\ i = S, W; \quad \Theta = \Psi(\theta). \tag{5.2}$$

Let R and B be the rank and bonus rewards in the non-linear constraint contest. Note that

$$\{X_i \geq \theta\} = \{Z_i = \Psi(X_i) \geq \Psi(\theta)\}, \quad \{X_i > X_j\} = \{Z_i = \Psi(X_i) > Z_j = \Psi(X_i)\}, \quad i = S, W. \tag{5.3}$$

Thus, the reward to manager $i = S, W$ from submitting Ψ-performance z in the Ψ-contest is

$$R\,\mathbb{P}[Z_j \leq z] + B\,\mathbb{1}_{\Theta}(z), \quad j = S, W, \text{ and } j \neq i, \tag{5.4}$$

and the capacity constraint is

$$\mathbb{E}[Z_i] \leq c_i, \quad i = S, W. \tag{5.5}$$

In these expressions, Z_i represents the Ψ-*performance* of a manager, i.e., performance measured by the amount of Ψ-capacity required to produce it.

Thus, a non-linear constraint contest, with capacity function Ψ, where managers choose performance (X_S, X_W) subject to the Ψ-capacity constraint given by equation (5.1), and face bonus threshold θ, corresponds to a Ψ-contest: a contest where managers choose Ψ-performance (Z_S, Z_W) subject to an expectational constraint (equation (5.5)) and face a bonus threshold of Θ. Given equilibrium performance in the Ψ-contest, equilibrium performance in the non-linear constraint contest, (X_S, X_W) is given by $X_i = \Psi^{-1}(Z_i)$, $i = S, W$, where Ψ^{-1} is the inverse of Ψ.

Because, in a Ψ-contest, the capacity constraint is expectational and the reward function is the same as the reward function in the baseline model, all of the results of our baseline model characterizing equilibrium performance, characterize equilibrium Ψ-performance in the Ψ-contest. To show our comparisons between mixed reward and bonus-reward contests also characterize non-linear constraint contests, we need only show that our characterizations of Ψ-performance, (Z_S, Z_W) in the Ψ-contest also characterize performance, (X_S, X_W) in the non-linear constraint contest.

First, consider ruin risk. In Proposition 2, we showed that ruin risk under mixed rewards is less than ruin risk under bonus or rank rewards. It is easy to see that these results are robust by noting that properties (a) and (b) imply that the ruin probability of Ψ-

performance, Z_i and performance in the non-linear constraint contest, X_i, are the same, i.e., $\mathbb{P}[X_i = 0] = \mathbb{P}[Z_i = 0]$.

Next, consider upside risk taking. The upper bound on performance in a non-linear constraint contest is a strictly increasing function of the upper bound on Ψ-performance in the corresponding Ψ-contest. Thus, our characterizations of the relationship between rank focus and the upper bound on performance in Section 4.2 are robust.

Finally, our stochastic order comparisons in Section 4.3 involved comparisons of performance under mixed rewards with performance under bonus rewards. By the correspondence between non-linear constraint contests and Ψ-contests, we know that these comparisons hold for Ψ-performance. To show that the results hold for performance in the corresponding non-linear constraint contest, (X_S, X_W), we need only show that if Z_1 and Z_2 are any two Ψ-performance distributions ordered by a stochastic order relation, then performance levels $X_1 = \Psi^{-1}(Z_1)$ and $X_2 = \Psi^{-1}(Z_2)$ in the corresponding non-linear constraint contest are ordered in the same direction.

The conditions for the preservation of these stochastic ordering relations are easy to identify. First note that if $\Psi' > 0$ and $\Psi'' > 0$, then $\Psi^{-1\prime} > 0$ and $\Psi^{-1\prime\prime} < 0$; if $\Psi' > 0$, $\Psi'' > 0$, and $\Psi''' < 3\,(\Psi'')^2/\Psi'$, then $\Psi^{-1\prime} > 0$, $\Psi^{-1\prime\prime} < 0$, and $\Psi^{-1\prime\prime\prime} > 0$ (eqs. 2 and 3: Apostol, 2000).

Proposition 1 in Denuit et al. (2013) shows that, if Z_1 second order stochastically dominates Z_2, $\Psi^{-1\prime} > 0$, and $\Psi^{-1\prime\prime} < 0$, then $\Psi^{-1}(Z_1) = X_1$ second-order stochastically dominates $\Psi^{-1}(Z_2) = X_2$. The same proposition shows that if $\Psi^{-1\prime} > 0$, $\Psi^{-1\prime\prime} < 0$, and $\Psi^{-1\prime\prime\prime} > 0$, then if Z_1 third-order stochastically dominates Z_2, $\Psi^{-1}(Z_1) = X_1$ third-order stochastically dominates $\Psi^{-1}(Z_2) = X_2$. Thus, under the noted restrictions, stochastic order relations between Ψ performance levels translate into stochastic order relations between performance levels in non-linear constraint contests.

In summary, the mollifying effects of mixed rewards on ruin risk-taking and upside risk taking generalize to all non-linear constraint contests. Our results verifying second-order stochastic dominance of performance under mixed rewards hold for any strictly increasing, strictly convex capacity function. Our results on third-order stochastic dominance of mixed rewards hold for any strictly increasing, strictly convex capacity functions, Ψ, satisfying $\Psi''' < 3\,(\Psi'')^2/\Psi'$, e.g., capacity functions that are convex power functions or exponential functions.

At the expense of even greater complexity, the analysis could be further extended to encompass more complex constraints. For example, instead of a single constraint defining feasibility, multiple constraints on the moments of the performance distributions could be imposed. Such constraints would, like the single non-linear constraint modeled above, clearly change the shape of the performance distribution, but not alter our basic results. Of course, restricting managers to choosing the scale of a common symmetric zero-mean risk distribution, i.e., forcing managers to accept downside risk when they want to take on upside risk and vice versa, would profoundly alter our conclusions. The standard result in these sort of risk-taking models of rank competitions (Hvide, 2002; Coles et al., 2020) is that weak managers will always choose infinite risk or, if there is an exogenous bound on the scale parameter, maximum risk.

We do not think that the dependence of our results on the assumption that managers can affect the shape of the performance distribution, say by varying skewness, is a drawback. We cannot identify any institutional constraints that force firm and fund performance to be symmetrically distributed. Nor do we find any evidence for such constraints in the data. Performance metrics for fund managers are frequently based on portfolio returns. There is a significant body of evidence showing that actively managed mutual funds, private equity firms, and hedge funds have skewed return distributions (e.g., Back et al., 2018; Brooks and Kat, 2002). Both the relative performance and bonus rewards for CEOs are, to a large extent, determined by accounting performance. The distributions of earnings and other accounting performance metrics exhibit "bunching," i.e., they are multi-modal because performance just below targets is rarely observed (Healy, 1985; Burgstahler and Dichev, 1997; Dharmapala, 2016).[12]

5.2 Linear and option compensation

5.2.1 Linear compensation

In our setting, linear compensation has no effect on risk taking incentives: under mixed rank and linear absolute performance rewards, managers will play the same strategies as they would in a rank competition contest. To see this, let γ represent the linear compensation coefficient. manager payoffs in the linear compensation setting would still be characterized by the support line conditions provided by Remark 3. Without loss of generality, take $r = 1$. If we let α', β' represent the multipliers for the support lines under linear compensation, the equilibrium best-reply conditions can be expressed as

$$\begin{aligned} &\forall x \in \mathrm{Supp}_S,\ \gamma\, x + F_W(x) = \alpha'_S + \beta'_S\, x, \qquad \forall x \geq 0,\ \gamma\, x + F_W(x) \leq \alpha'_S + \beta'_S\, x; \\ &\forall x \in \mathrm{Supp}_W,\ \gamma\, x + F_S(x) = \alpha'_W + \beta'_W\, x, \quad \forall x \geq 0,\ \gamma\, x + F_S(x) \leq \alpha'_W + \beta'_W\, x. \end{aligned} \tag{5.6}$$

If we let $\alpha'_i = \alpha_i$ and $\beta'_i = \beta_i + \gamma$, $i = S, W$, where α_i and β_i represent the multipliers for the Eq0 equilibrium, we see that the distribution functions specified in Lemma 4 satisfy the equilibrium conditions given by equation (5.6). Thus, the Eq0 equilibrium distribution functions are equilibrium distribution functions under linear compensation.

[12]See Kleven (2016) for a survey of methodological issues in this literature.

5.2.2 Option-based compensation

In the baseline model, we studied absolute performance rewards that took form of bonuses attained if performance at least equaled a fixed bonus threshold. This raises the question of the extent to which our results depend on this absolute-performance reward specification. To address this question, we extend our analysis to consider the most prevalent alternative to bonus rewards: option-based rewards.

A complete analysis of the option setting is beyond the scope of this paper. In order to provide some insight into how rank rewards interact with option rewards as well as the similarities and differences between the bonus and option settings, we will sketch the characterization of Eq1 equilibria in the option setting. Our sketch will focus on the steps in the development that are unique to the option setting. Appendix Section B will present the remaining details as well as characterizations of Eq2 configurations under option compensation.

In an Eq1 equilibrium in the baseline bonus setting, W randomizes between 0 performance (with probability p_W^0) and uniformly distributed performance (with probability $1-p_W^0$) over an interval whose lower bound is 0 and upper bound is less than the minimum performance required to capture the absolute performance reward. S also randomizes over the same range of performance as W but also devotes some capacity to targeting the absolute performance reward, i.e., places some mass on single performance level that will capture the absolute performance reward. We will show that, under option compensation, Eq1 configurations can also be supported and provide an example of an Eq1 equilibrium.

We assume that managers who submit performance x receive an absolute performance reward equal to $\max[x-\theta, 0]$, where $\theta > 0$. Thus, absolute rewards take the form of a European call option with a strike price of θ.[13] Other than this change in the form of the absolute performance reward, the objective functions of the managers is the same as specified in the baseline bonus setting (equation (2.2)). For simplicity, we assume that the rank reward, R, equals 1.

Thus, the reward functions of S and W, Π_S and Π_W respectively, under option rewards are given by

$$\begin{aligned}\Pi_S(x) &:= F_W(x) + \max[x-\theta, 0],\\ \Pi_W(x) &:= F_S(x) + \max[x-\theta, 0].\end{aligned} \tag{5.7}$$

Comparing equation (5.7) with the reward function in the baseline bonus setting (equation (2.1)), shows that modifying the reward function to incorporate option rewards is straightforward. Unfortunately, modifying the capacity constraint to accommodate option

[13]Using θ to represent the strike price of the option is to some degree an abuse of notation as θ in the baseline model represents a bonus threshold. We adopt this notation to avoid the needless introduction of new notation. Because we only consider option compensation in this section, this notation should not cause any confusion.

rewards is not. The capacity constraint in the bonus setting only constrains expected performance, the convexity of the option claim implies that, even in the absence of rank competition, under pure expectational constraints, risk taking is unbounded.

Consequently, in order for equilibrium performance to be defined under option rewards, some bound on the higher moments of the distribution is required. We define a *mixed contest* in the option setting by postulating the simplest effective bound on the higher moments, a *quadratic constraint*

$$\mathbb{E}[X_i^2] \leq c_i, \quad i = S, W, \quad c_S > c_W > 0.$$

Under this constraint, the mixed contest is a non-linear constraint contest of the sort analyzed in Section 5.1. Equilibrium behavior in the mixed contest can be determined using the characterizations in baseline setting and the approach developed in Section 5.1: transform the mixed contest with a quadratic constraint on performance into an *sq[uare]-contest* where managers submit *sq-performance*, $Z_i = X_i^2$ and the capacity constraint on performance is expectational, i.e.,

$$\mathbb{E}[Z_i] \leq c_i. \tag{5.8}$$

After solving for the equilibrium sq-performance distributions, find equilibrium performance in the mixed contest by transforming sq-performance (Z_S, Z_W) into performance using the relationship between sq-performance and performance: $(X_S, X_W) = (\sqrt{Z_S}, \sqrt{Z_W})$.

The reward functions in the sq-contest for S and W respectively are

$$\Pi_S^{\mathrm{sq}}(z) := F_W^{\mathrm{sq}}(z) + \max[\sqrt{z} - \theta, 0], \tag{5.9}$$

$$\Pi_W^{\mathrm{sq}}(z) := F_S^{\mathrm{sq}}(z) + \max[\sqrt{z} - \theta, 0], \tag{5.10}$$

where F_i^{sq} represents the distribution function of $Z_i = X_i^2$. The capacity constraints are given by equation (5.8). Except for the functional form of the absolute performance reward, the sq-contest is formally identical to the sort of expectational contests we analyzed in the baseline bonus setting. Thus, we can use the baseline characterizations, *mutatis mutandis*, to characterize equilibrium performance in the sq-contest.

To simplify notation, define

$$\Delta_S(z) = \Pi_S^{\mathrm{sq}}(z) - (\alpha_S + \beta_S\, z), \quad z \geq 0. \tag{5.11}$$

As shown in Section 2.2, the necessary and sufficient conditions for an S-performance distribution being a best response are

$$\text{For all } z \in \mathrm{Supp}(F_S^{\mathrm{sq}}),\ \Delta_S(z) = 0, \tag{5.12}$$

$$\text{For all } z \geq 0,\ \Delta_S(z) \leq 0. \tag{5.13}$$

In the sq-contest, we represent the upper bound on W's sq-performance by v. If an Eq1 equilibrium for the sq-contest exists, the sq-performance distribution for W in this equilibrium will equal

$$F_W^{\mathrm{sq}}(z) = p_W^0 + (1 - p_W^0) \min\left[\frac{z}{v}, 1\right], \quad v \in [0, \theta^2). \tag{5.14}$$

Because, in an Eq1 configuration, S also randomizes uniformly over $[0, v]$ with positive probability, $[0, v] \subset \mathrm{Supp}_S$. For this reason, equation (5.12) implies that

$$\alpha_S = p_W^0, \quad \beta_S = \frac{1 - p_W^0}{v}. \tag{5.15}$$

In an Eq1 configuration, S, and only S, targets absolute performance rewards. In the option setting, these rewards can be attained only if sq-performance, z, exceeds θ^2. Because, in an Eq1 configuration, the upper bound of W's performance is insufficient to capture absolute performance rewards, $\theta^2 > v$. Thus, performance that captures an absolute performance reward captures the rank reward (which was assumed to equal 1) with certainty. Hence,

$$\Delta_S(z) := (1 + (\sqrt{z} - k)) - (\alpha_S + \beta_S\, z), \quad z > \theta^2. \tag{5.16}$$

Thus, for $z > \theta^2$, Δ_S is strictly quasiconcave. Hence, there is at most one maximizer of Δ_S over $z > \theta^2$. If no maximizer exists, then equation (5.12) implies that no $z > \theta^2$ is in the support of F_S^{sq}, i.e., an Eq1 configuration cannot be supported by equilibrium performance distributions. Simple calculus shows that if an interior maximizer, z_o, of Δ_S exists,

$$z_o = \frac{1}{4\beta_S^2} \text{ and } z_o > \theta^2. \tag{5.17}$$

Equations (5.12) and (5.13) can be satisfied only if

$$\Delta_S(z^o) = 0, \tag{5.18}$$

$$\text{For all } z > \theta^2,\ \Delta_S(z) \le 0. \tag{5.19}$$

Strict quasiconcavity of Π_S^{sq} over $z > \theta^2$ implies that (5.19) is satisfied as a strict inequality for $z \neq z_o$. Thus, z_o is the only point in the support of F_S^{sq} that exceeds v, the upper bound for W's sq-performance. We term z_o the *sq-option target*.

Hence in an Eq1 equilibrium in the option setting, as in the baseline setting, sq-performance by S (performance in the baseline setting) in excess of the sq-performance level of W will be targeted on a single point, the *sq-option target*. The difference between the option and bonus settings is that, in the option setting, the target is endogenously determined rather than being exogenously fixed by the bonus threshold.

Because of endogenous targeting, in the option setting, characterizations of manager behavior are much less transparent than in the bonus setting. Applying equations (5.15), (5.17), (5.18), and the sq-capacity constraint for W in an Eq1 equilibrium, does yield an equation that implicitly defines v, the upper bound on rank competition in the sq-contest, as the root of a cubic equation. Using Cardano's formula, this root can be determined in closed form. However, in general, the root can only be expressed through compositions of trigonometric and inverse trigonometric functions. In contrast, the upper bound for W's performance under bonus compensation is a simple radical expression.

With v determined, characterizations of the other parameters follow easily through imposing the sq-capacity constraints. The characterizations of the performance in the mixed contest then follow from transforming restrictions on sq-performance in the sq-contest into restrictions on performance in the mixed contest using the identity, $(X_S, X_W) = (\sqrt{Z_S}, \sqrt{Z_W})$. For example, the sq-option target for S, z_o, maps into the *option target* $x_o = \sqrt{z_o}$ in the mixed contest.

To provide further insight into the differences and similarities between the bonus and option settings, we judiciously select parameters for the option contest to ensure that the solutions for the mixed contest can be expressed in closed form without using trigonometric functions. The parameter utilized are $c_W = 1$, $c_S = 1\frac{37}{40}$, and $\theta = 1\frac{19}{24}$.

The equilibrium performance distributions for S and W in the mixed contest are given below:

$$F_W(x) = p_W^0 + (1 - p_W^0) \min\left[\left(\frac{x}{u}\right)^2, 1\right], \quad x \geq 0,$$
$$F_S(x) = (1 - p_S^{x_o}) \min\left[\left(\frac{x}{u}\right)^2, 1\right] + p_S^{x_o} \mathbb{1}_{x_o}(x), \quad x \geq 0,$$
$$x_o = 2\tfrac{1}{4},\ u = \sqrt{v} = \sqrt{3},\ p_W^0 = \tfrac{1}{3},\ p_S^{x_o} = \tfrac{34}{285},$$

where $p_S^{x_o}$ represents the probability weight placed by S on the option target. The statistics for equilibrium performance in the option setting, under option, rank, and mixed rewards are provided in Table 2.

	Rewards		Ruin Risk		Option Target		Length(Support)	
	Option	Rank	$\mathbb{P}[X_S = 0]$	$\mathbb{P}[X_W = 0]$	$x_o(S)$	$x_o(W)$	$\lambda[\text{Supp}_S]$	$\lambda[\text{Supp}_W]$
Option contest*	$\theta = 1.79$	0	0.85	0.92	3.58	3.58	0.00	0.00
Mixed contest	$\theta = 1.79$	1	0.00	$0.3\bar{3}$	2.25	None	1.73	1.73
Rank contest	None	1	0.00	0.48	None	None	1.96	1.96
Parameters	$c_W = 1$, $c_S = 1\,\frac{37}{40}$, and $\theta = 1\,\frac{19}{24}$							

* Equilibrium performance distributions in the option contest are computed in Appendix Section B.

Table 2: Statistics for manager performance in the option setting.

As Table 2 reveals, the qualitative effects of mixed rewards under option and bonus compensation are quite similar. As in the baseline bonus setting, introducing rank rewards leads managers to spread out performance. Spreading performance reduces ruin risk. As in the bonus setting, there is also a countervailing effect of rank rewards on risk taking: increasing rank rewards decreases the allocation of S-capacity to targeting the option reward. This effect makes S a more effective rank-competitor and thus leads to an increase in the effective strength asymmetry between the weak and strong manager. Because strength asymmetry is the source of ruin risk taking in rank competitions, the reduced option targeting by S increases ruin risk taking. Thus, as in the bonus setting, under mixed incentives, ruin risk taking in the mixed contest is less than in either the rank or absolute performance contest. Performance distributions for S and W under mixed rewards are plotted in Figure 8.

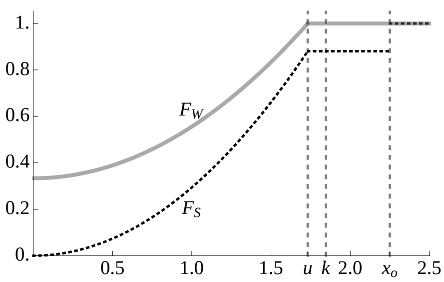


Figure 8: The equilibrium performance distributions. In the figure F_S and F_W, represent equilibrium performance distributions in an Eq1 equilibrium. The parameters used to draw the figure are provided in Table 2.

In the option setting, there is a novel channel through which mixed rewards reduce risk taking. In a mixed contest, the option target, x_o, is lower than the option target in the absence of rank-rewards. In the bonus setting, the absolute-reward target is fixed. Another novel feature of mixed rewards in the option setting, illustrated in Figure 8 and discussed earlier (Section 2.3), is that, because of the non-linear capacity constraint, performance distributions do not have any uniform components.

5.3 Assured bonus compensation

The baseline model assumes that bonus rewards are *unassured*, i.e., contestants do not have sufficient capacity to capture bonus rewards with certainty. Thus, capturing bonus rewards requires risk taking. We do not believe that the assumption that bonus rewards are unassured is a significant limitation because, (i) the incentives produced by "high-powered," and hence unassured, rewards have been the focus of most research on (absolute) performance incentives, and (ii) unassured rewards are pervasive. Even CEOs, whose bonus targets are notoriously unambitious, miss targets about 40% of the time (Kay et al., 2015).

Nevertheless, exploring the implications of assured rewards is worthwhile because of the insight it provides into the logic underlying our analysis. Our basic result is that if bonus rewards are assured, that thus bonus compensation does not generate an incentive to gamble, introducing rank rewards will never reduce risk-taking and may sometimes increase risk taking.

We formally develop this rather obvious result in Appendix Section C. Here we only sketch our argument: if bonus rewards are assured for both the strong and weak contestant, i.e., $\theta < \mu_W$, in the absence of rank competition, both S and W will always capture the bonus reward with probability 1. Thus, neither S nor W will assume any ruin risk; clearly, in this case, ruin risk cannot be reduced by introducing rank incentives because there is no risk taking to reduce.

If S's but not W's bonus reward is assured, i.e., $\mu_W < \theta < \mu_S$, then, absent rank rewards, W will assume ruin risk and S will not. Under mixed rewards, as under bonus rewards, S will never accept ruin risk. Depending on the degree of strength asymmetry, μ_S/μ_W, and rank focus parameter, r, W will either (i) concede rank dominance to S and focus on capturing the bonus, in which case rank-rewards will have no affect on ruin risk taking as they will not change W's equilibrium strategy or (ii) W will compete with S for rank dominance. In this case, because S's capacity exceeds the bonus threshold, rank competition with S will require accepting even *more* ruin risk than bonus targeting. Thus, the introduction of rank rewards will increase ruin risk.

5.4 Other extensions

5.4.1 Different bonus thresholds

In the baseline model, we assume that both managers face the same bonus threshold. As we show in Appendix Section D, if managers face different thresholds, and the threshold for the weak manager is sufficiently lower than the threshold for the strong manager, a novel equilibrium configuration emerges: the weak and strong managers compete for rank dominance only at performance levels below both bonus thresholds, each manager also targets his/her own bonus threshold.

As in the baseline model, when bonus thresholds differ across managers, mixed rewards mollify risk-taking incentives in this configuration as well. The intuition for this mollifying effect is also the same as the intuition developed in the baseline model. Rank rewards lead the managers to spread out their performance and thereby reduce ruin risk taking relative to bonus competition. At the same time, as illustrated in the example, when bonus rewards motivate bonus chasing, the strong manager diverts more capacity to bonus chasing. This makes the strong manager a less effective rank competitor. This effect ensures that, under mixed rewards, the ruin risk the weak manager must accept to compete with the strong manager for rank dominance in the subthreshold region (for both managers) is less than ruin risk the weak manager must accept under rank rewards.

5.4.2 Multiple managers

In Appendix Section E, we show that in the same basic characterizations of managerial incentives in bonus, rank, and mixed contests that we developed for two manager contests, also characterize multi-manager contests. As in two-manager contests, spreading out performance is efficient for capturing rank rewards and, in rank contests, the marginal gain from increased capacity is smaller for stronger contestants. Thus, there is no reason to suspect that simply adding more managers will lead to different qualitative conclusions than the baseline model. Solving the general problem of characterizing equilibrium behavior in all asymmetric multi-manager rank-and-bonus risk taking contests is a worthy objective. But it is not a trivial task and is outside the scope of this paper. However, as we illustrate by an example in Appendix Section E, it is easy to numerically construct multi-manager contest equilibria in which mixed competitions produce less ruin risk taking than both bonus and rank competitions.

6 Implications

The central insight of this paper is that mixed rewards lead to "strategy jamming." Risk taking strategies that are efficient for capturing absolute performance rewards are very inefficient for capturing rank rewards and vice versa. Thus, when rewards are mixed, the risk-taking incentives produced by absolute and relative performance rewards tend to "cancel" each other out. In Section 5 we showed that this cancellation effect is founded on very generic properties of rank competition.

A transparent implication of strategy jamming, and the implication we have emphasized in this paper, is that managers, motivated by a mixture of rank and absolute performance rewards, have a much smaller appetite for risk taking than models based only on compensation incentives or tournament rewards predict. However, strategy jamming also produces a number of other novel insights into managerial risk taking behavior.

Effect of exogenous shocks on risk taking behavior Consider a scenario in which, given expectations at the time of contracting, performance rewards are moderately challenging and that between-manager strength asymmetries are not extreme (as defined in Proposition 4). If an unexpected economic shock occurs that reduces the capacity of managers to generate returns, how will the shock affect risk taking?

Since, pre-shock rewards are moderately challenging and strength asymmetries are not extreme, given pre-shock capacity, the equilibrium configuration is Eq2 (Proposition 4), i.e., both strong and weak managers chase absolute performance rewards. A shock, s, that reduces performance capacity from (μ_S, μ_W) to $(\mu_S/s, \mu_W/s)$, $s > 1$ is equivalent to increasing the bonus threshold from θ to $s\,\theta$. If the shock is small, i.e., s is not much larger than

1, the post-shock equilibrium configuration will also be Eq2. Lemma 6 shows that, in Eq2 configurations, increasing θ reduces ruin risk. Thus, ruin risk taking will be reduced by small adverse shock to the managers' performance capacity.

In contrast, if the adverse shock is severe, reduced performance capacity will make reaching high water marks or bonus thresholds nearly impossible and an Eq0 configuration will emerge. Strong managers will focus solely on besting weak managers in rank competition. The increased focus of strong managers on rank competition will increase ruin risk taking (Proposition 2).

Although, to our knowledge, this implication of our model has not been tested, the effect of exogenous shocks on managerial risk taking has been considered in a number of empirical studies. For example, Hayes et al. (2012) use a quasi-natural experiment, the adoption of FAS123R, to evaluate the effect of managerial compensation on risk-taking behavior. They find that the implementation of FAS123R, which increased the cost of offering option compensation to managers, did not reduce managerial risk taking. In our framework, this result is not surprising: reduced absolute performance rewards increase managers' focus on labor market tournament rewards and thus can increase tournament-motivated risk taking.

Cross-sectional implications of mixed rewards The extensive body of empirical research on tournament incentives and risk taking by financial professionals is motivated, to a very large extent, by theoretical models of rank competitions. The motivating theoretical literature models agents who do not receive any rewards for absolute performance, and thus does not approximate, even in a first-order sense, the structure of incentives for financial managers.

Our analysis provides many new testable implications for such research. For the sake of brevity, we will outline only a few, focusing on the arguably most empirically plausible case: strength asymmetry that is "moderate" (as defined in Proposition 4). First consider the effect of varying the bonus threshold. Provided that the bonus threshold is low enough to support an Eq2 equilibrium, Lemma 6 shows that increasing the bonus threshold *reduces* ruin risk taking. If the threshold increases sufficiently the equilibrium configuration switches to Eq1, and, as shown in Lemma 6, in Eq1 configurations, increasing the threshold increases ruin-risk taking. Thus, the relationship between ruin risk taking and the bonus threshold is U-shaped. Fixing the other parameters of the model, the magnitude of the bonus threshold measures "target difficulty." Chen et al. (2019) using a linear specification, find no empirical relationship between target difficulty and risk-taking. This result would be surprising in a world where all performance rewards are generated by compensation tied to absolute performance levels. In this world, the risks that must be incurred in order to attain a higher target will be greater than the risk required to reach a more modest target. In our setting, the lack of a linear relationship between target difficulty and risk taking is not surprising because in our setting, where rank incentives are important, there is no linear or even monotone relationship between risk taking and target difficulty.

If we consider the effect of rank rewards on upside risk taking, we see that when bonus threshold are moderately challenging (see Proposition 4), the relation between rank focus and upside risk is inverse U-shaped and the lower bound on upside performance is the bonus target. The bound is approached both when rank focus is very small and when rank focus is very high. Thus, the relationship between upside risk taking and rank-focus is roughly inverse U-shaped, and upside risk taking is greatest when rank and bonus rewards are roughly balanced. Although finding proxies for rank-focus is admittedly more difficult than measuring target difficulty, in principle, proxy variables for fund-flow sensitivity to ranking (for fund managers) or career concerns (for CEOs and top executives) might be employed. Thus, this implication is testable, although, to our knowledge, untested.

Identifying the strength of rank dominance preferences in experiments In a laboratory experiment, Kirchler et al. (2018) document that the risk-taking behavior of financial professionals is influenced by rankings, even if rankings are not associated with pecuniary rewards or publicly observable. Given this evidence, a natural question to ask is how much wealth are financial professionals willing to sacrifice to attain rank dominance? As shown in Section 2.1, when pecuniary rewards are non-linear in performance, the pursuit of rank dominance has opportunity costs: rank efficient performance strategies are not efficient strategies for capturing absolute performance rewards. By using our analysis of optimal performance strategies in mixed contests, and carefully choosing the functional form of absolute performance rewards, it is possible to ensure that the map between r, the rank focus parameter, and risk taking strategies is 1-1. Thus, an experiment on financial professionals could identify the intensity of subjects' rank-dominance preferences from observed risk-taking strategies.

Effect of rank incentives on the bunching of CEO performance metrics As discussed in the introduction, the distributions performance metrics for CEOs, e.g., earnings, frequently exhibit "bunching," with performance concentrated just above apparent target levels. Target levels that accounting research has linked to bonus thresholds (Healy, 1985; Burgstahler and Dichev, 1997). In our theoretical analysis, in Eq2 equilibria, both managers bunch performance just above bonus threshold. The degree to which performance is bunched is inversely related to the length of the superthreshold competition region, $u_H - \theta$.

Proposition 4 shows that when the rank focus parameter, r, is small, and thus bonus rewards are large relative to rank rewards, performance is concentrated in the region just above the bonus target. Thus, our analysis predicts that the performance of CEOs who have general human capital, and are incentivized primarily by rank rewards from labor market competition, will bunch performance less than managers with task or firm-specific human capital who are motivated primarily by contracted bonus rewards.

Identifying managerial ability with censored samples of performance Is it possible to identify the competences of competing managers before a ruin event, $X_i = 0$, $i = S, W$ is realized? Of course, to an econometrician studying a survivorship-bias free sample of hedge funds, the answer to this question is of little import. However, an under-diversified hedge-fund investor would probably find the answer more interesting.

In both bonus and rank competitions, the answer to this question is no, as evidenced by Result 1 and Lemma 4, the distributions of S and W managerial performance censored at 0, which we denote by $[X_i|X_i > 0]$, $i = S, W$, are identical. In contrast, in mixed competitions, it is always possible to identify ability from censored performance.

Result 2. (i) If the equilibrium configuration is Eq2, then $[X_S|X_S > 0]$ second-order stochastically dominates $[X_W|X_W > 0]$. (ii) If the equilibrium configuration is Eq1, then $[X_S|X_S > 0]$ first-order stochastically dominates $[X_W|X_W > 0]$.

Thus, when rewards are mixed, strong and weak managers are observationally different even if observed performance does not contain a ruin event: the performance of W is always riskier in the sense of Rothschild and Stiglitz (1970) than the performance of S. Consequently, even when investors are not concerned about the unsystematic risk produced by fund-manager gambling, when managers receive mixed rewards, it is sensible to evaluate manager ability using metrics that incorporate unsystematic risk.

7 Conclusion

In this paper, we analyzed the interaction between rank-based rewards, and rewards conditioned on attaining an absolute level of performance. Both rank and absolute performance rewards encourage risk taking. However, we showed that introducing rank rewards into absolute performance competitions, or introducing absolute performance rewards into rank competitions, always reduces managerial ruin risk taking and that, under quite general conditions, reduces the overall riskiness of managerial performance. When managers are motivated by rank dominance as well as absolute performance rewards, the relationships between rewards and risk taking are quite different from, and sometimes directly opposed to, the relationships generated by either pure rank or pure absolute performance rewards.

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Appendix:
The golden mean: The risk mitigating effect of combining tournament rewards with high-powered incentives

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Contents

A Proofs of results in Sections 2–4

Proof of Proposition 1 To prove (iii), simply note that if expected performance under manager i's expected equilibrium performance, F_i^*, say μ_i^*, was less than μ_i, because $\mu_i < \theta$, the manager would not be capturing the bonus with certainty. Thus, the manager could improve on his equilibrium payoff by using excess capacity, $\mu_i - \mu_i^* > 0$, to target the bonus reward by placing some positive probability mass on θ, contradicting the optimality of F_i^*.

We only prove parts (i) and (ii) the proposition for $i = W$. The proof for $i = S$ is identical save for the transposition of S and W. First note that we can express F_W as the sum of a purely discontinuous distribution, $F_{W,d}$, and a continuous distribution, $F_{W,c}$, i.e.,

$$F_W(x) = F_{W,d}(x) + F_{W,c}(x), \quad x \in \mathbb{R}.$$

First, consider part (i). Define the following three sets:

$$\begin{aligned} \mathscr{N} &= \{x \in \mathbb{R} : x < 0\} \\ \mathscr{Z}_W &= \{x \in \mathbb{R}^+ : x \notin \mathrm{Supp}_{W,c}\} \\ \mathscr{S}_W &= \{x \in \mathbb{R}^+ : x \neq \theta \text{ and } x \in \mathrm{Supp}_{W,c}\} \end{aligned} \tag{A-1}$$

Note that $\mathbb{R} = \mathscr{N} \cup \mathscr{S}_W \cup \mathscr{I}_W \cup \{\theta\}$. If $x \in \mathscr{N}$ then because, by assumption, performance distributions are non-negative, so $F_W'(x) = 0$. If $x \in \mathscr{Z}_W$, then by the definition of the support of a distribution function, $F_W'(x) = 0$.

Now consider $x \in \mathscr{S}_W$. Note that, because, $F_{W,c}$ is continuous, $\mathscr{S}_W$ contains no isolated points. Let $(y_n) \in \mathscr{S}_W$ be a sequence of points converging to x.

By Remark 2.a, $\mathscr{S}_W \subseteq \mathrm{Supp}_S$. Thus, x and y_n, for all n, are in Supp_S. Because, x and y_n are in Supp_S, the optimality condition (Remark 3.ii) implies that $\Pi_S(x) = \ell_S(x)$ and $\Pi_S(y_n) = \ell_S(y_n)$. For this reason,

$$\frac{\Pi_S(y_n) - \Pi_S(x)}{y_n - x} = \frac{\ell_S(y_n) - \ell_S(x)}{y_n - x} = \beta_S. \tag{A-2}$$

Because $y_n \to x$ and $x \neq \theta$, for n sufficiently large, either $y_n > \theta$ or $y_n < \theta$. Thus, for n sufficiently large, either $x > \theta$ and $y_n > \theta$ or $0 \le x < \theta$ and $0 \le y_n < \theta$. Thus, by Remark 2.c, no jumps occur between y_n and θ, which implies that $F_{W,c}(y_n) - F_{W,c}(x) = F_W(y_n) - F_W(x)$.

For this reason, the definition of the reward function (equation (2.6)), implies that

$$\frac{\Pi_S(y_n) - \Pi_S(x)}{y_n - x} = \frac{r\, F_{W,c}(y_n) - r\, F_{W,c}(x)}{y_n - x}. \tag{A-3}$$

Equations (A-2) and (A-3) thus imply the following result:

Result A.1. For every $x \in \mathscr{S}_W$, there exists a sequence, (y_n) such that $y_n \to x$ and

$$\frac{F_{W,c}(y_n) - F_{W,c}(x)}{y_n - x} \to \frac{\beta_S}{r}.$$

Result A.1 implies that, for $x \in \mathscr{S}_W$, if $F'_{W,c}(x) \neq \beta_S/r$ then $F'_{W,c}(x)$ does not exist. Thus,

$$\{x \in \mathbb{R} : F'_{W,c}(x) \neq 0 \text{ and } F'_{W,c} \neq \beta_S/r\} \subseteq \{x \in \mathbb{R} : F'_{W,c}(x) \text{ does not exist}\} \cup \{\theta\}. \quad \text{(A-4)}$$

Lebesgue's theorem for the differentiability of monotone functions shows that $\{x \in \mathbb{R} : F'_{W,c}(x) \text{ does not exist}\}$ has measure 0. Obviously the single point θ has measure 0. Since a monotone function as a most a countable number of jumps, and countable sets have 0 measure, for almost all $x \in \mathbb{R}$, $F'_{W,c}(x) = F_W(x)$. This proves part (i) of the proposition.

To prove, part (ii), note that Remark 2.c implies that the discontinuous component takes the form specified in the proposition: $F_{W,d} = \gamma_0 \, \mathbb{1}_0(x) + \gamma_\theta \, \mathbb{1}_\theta(x)$. Thus,

$$F_i(x) = \gamma_0 \, \mathbb{1}_0(x) + \gamma_\theta \, \mathbb{1}_\theta(x) + F_{W,c}(x). \quad \text{(A-5)}$$

Now consider the continuous component, $F_{W,c}$. All we need to show is that $F_{W,c}$ has no singular component. This is straightforward. Vallée Poussin's Theorem implies that a necessary condition for $F_{W,c}$ to have a singular component is that, on an uncountable (but Lebesgue measure 0) subset of $\mathbb{R}$, $F'_{W,c} = \infty$ (pp. 482 Russell, 1979). However, our analysis thus far has shown that the only point at which it might be the case that $F'_{W,c} = \infty$ is $x = \theta$. Hence, $F_{W,c}$ is absolutely continuous. Absolute continuity implies that

$$F_{W,c}(x) = \int_{-\infty}^{x} F'_{W,c}(x)\, dx = \int_{0}^{x} F'_{W,c}(x)\, dx. \quad \text{(A-6)}$$

Equations (A-5) and (A-6) imply the characterization in the proposition.

Proof of Lemma 1 The following technical lemmas greatly simplify the derivations in the proof of this lemma.

Lemma A.1 *In an Eq2 equilibrium, the supports of both managers' (S and W) performance distributions are identical and there exist u_L and u_H such that $u_L < \theta < u_H < \infty$ and* $\mathrm{Supp}_W = \mathrm{Supp}_S = [0, u_L] \cup [\theta, u_H]$.

PROOF: By hypothesis, both S and W chase the bonus, so $F_S(\theta-) < 1$ and $F_W(\theta-) < 1$. By Remark 1, it is not possible for both S and W to place positive mass on the bonus threshold, θ, for this would result in tied performance. Thus, the supports of at least one of the managers' performance distribution must contain some point strictly greater than θ.

Because performance distributions must be continuous and connected except perhaps at 0 and θ (Remark 2.(c) and Remark 2.(b)), the support of one of the managers' performance distributions must be an interval. Thus, because the supports are the same for $\theta > 0$ (Remark 2.(a)), and because, by definition, supports are closed sets, the supports of the two managers' performance distribution in the interval $[\theta, \infty)$ must be a common interval.

The lower bound of this interval must be θ. Otherwise, by the continuity of the reward function above θ, which is implied by the continuity of the performance distributions above θ, performance equal to θ would produce the same reward as performance equal to the lower bound and use more capacity. In which case, the lower bound would lie below the managers' support lines. Hence, performance for both managers, for $x \geq \theta$, is supported by interval whose lower bound is θ.

From the definitions of the reward functions for the managers (equation (2.1)) for $x \geq \theta$, and from the fact that all points in the support of the managers' performance distributions lie on their respective support lines,

$$\begin{aligned} \forall x \in \mathrm{Supp}_W \cap [\theta, \infty),\ \Pi_W(x) = \ell_W(x) = \alpha_W + \beta_W\, x = 1 + r\, F_S(x), \\ \forall x \in \mathrm{Supp}_S \cap [\theta, \infty),\ \Pi_S(x) = \ell_S(x) = \alpha_S + \beta_S\, x = 1 + r\, F_W(x). \end{aligned} \tag{A-7}$$

Next, note that these equations can only be satisfied if F_S and F_W are affine on their supports for $x > \theta$. This implies that the performance distributions have an upper bound, say u_H; otherwise, F_S and F_W would be unbounded which is not possible given that they are distribution functions and thus bounded by 1.

Using a very similar argument it can be shown, using Remark 2, that, in the subthreshold region, the supports of the two managers' distributions are intervals whose lower bound equals 0 and an upper bound equals say u_L. By the definition of the subthreshold region, $u_L \leq \theta$. In fact, $u_L < \theta$ because the reward function jumps up at θ. Thus performance in a sufficiently small left neighborhood of θ produces a smaller reward than performance equal to θ.

Thus, for all $x < \theta$ the supports of performance distributions of S and W are the same and,

$$\begin{aligned} \forall x \in \mathrm{Supp}_W \cap [0, \theta),\ \Pi_W(x) = \ell_W(x) = \alpha_W + \beta_W\, x = r\, F_S(x), \\ \forall x \in \mathrm{Supp}_S \cap [0, \theta),\ \Pi_S(x) = \ell_S(x) = \alpha_S + \beta_S\, x = r\, F_W(x). \end{aligned} \tag{A-8}$$

□

Lemma A.2 *In an Eq2 equilibrium, the support lines of the two managers, S and W, have the following properties: $\alpha_S > 0$, $\alpha_W = 0$, $\beta_S > 0$, and $\beta_W > 0$, and $\beta_S < \beta_W$.*

PROOF: The fact that β_S and β_W are positive follows simply from the fact that the marginal value of capacity, which can always be applied to either increasing the probability of winning the rank reward or capturing the bonus, is positive.

Now consider α_S and α_W. Note that it cannot be the case that both α_S and α_W are positive. Over the subthreshold region, the reward to each manager from performance x equals the distribution function selected by the other manager evaluated at x. Let $\ell_i(x) = \alpha_i + \beta_i\, x$, $i = S, W$ represent the support lines of the managers. Because 0 is in the support of both managers' performance distributions (Remark 2.(d)), at point x in the support of the performance distribution, the reward to a manager from performance x lies on the manager's support line, i.e., $\ell_i(0) = \Pi_i(0) = \alpha_i$, $i = S, W$ (Remark 3). Thus, if both α_S and α_W were positive, $\Pi_S(0) > 0$ and $\Pi_W(0) > 0$. Since $\Pi_S(0) = r\, F_W(0)$ and $\Pi_W(0) = r\, F_S(0)$, this would imply that there is a positive probability of tied performance, which is impossible (Remark 1). So, if it is not the case that $\alpha_S > 0$ and $\alpha_W = 0$, then $\alpha_S = 0$ and $\alpha_W \geq 0$.

So to obtain a contradiction, suppose that $\alpha_S = 0$ and $\alpha_W \geq 0$. Next, note that, at u_H, the upper bound of the common support of F_S and F_W, F_S and F_W equal 1 and thus, from equation (A-7) we see that

$$1 + r = \ell_W(u_H) := \alpha_W + \beta_W\, u_H = \ell_S(u_H) := \alpha_S + \beta_S\, u_H = \beta_S\, u_H. \tag{A-9}$$

So,

$$\ell_W(u_H) = \ell_S(u_H). \tag{A-10}$$

At $x = 0$, because by the hypothesis being contradicted, $\alpha_S = 0$ and $\alpha_W \geq 0$, we see from equation (A-8), that at $x = 0$

$$\ell_W(0) \geq \ell_S(0). \tag{A-11}$$

Because the support lines are lines, and thus cross only once (or are identical), equations (A-11) and (A-10) imply that, for all $x \leq u_H$, $\ell_W(x) \geq \ell_S(x)$. Inspecting equations (A-7) and (A-8) and noting that the upper bound on managers' performance distributions is u_H, shows that these equations can only be satisfied if, for all x in the common support of the managers' performance distributions, $F_S \geq F_W$, which, because F_S and F_W are constant off of their common support, implies that for all $x \geq 0$, $F_S \geq F_W$, i.e., F_S is first-order stochastically dominated by F_W. This is not possible because the mean performance of S equals $\mu_S > \mu_W$, the mean performance of W. This contradiction implies that in all Eq2 equilibria, $\alpha_S > 0$, $\alpha_W = 0$. Equation (A-9), and $\alpha_S > 0$, $\alpha_W = 0$, imply that $\beta_S < \beta_W$. □

Let Supp represent the common support of the two managers' performance distributions (Lemma A.1). The fact that the support lines, ℓ_S and ℓ_W cross only once at $x = u_H$ (equation (A-9), and Lemma A.2) and $\ell_S(0) = \alpha_S > \ell_W(0) = \alpha_W = 0$, imply that, if $x < u_H$, $\ell_S(x) > \ell_W(x)$.

Remark 3 and the definition of the reward functions imply that, for all $x \in$ Supp,

$$\begin{aligned}\ell_W(x) = \beta_W\, x = \Pi_W(x) = r\, F_S(x) + \mathbb{1}_\theta(x),\\ \ell_S(x) = \alpha_S + \beta_S\, x = \Pi_S(x) = r\, F_W(x) + \mathbb{1}_\theta(x).\end{aligned}$$

Thus,

$$F_S(x) < F_W(x), \quad x \in \text{Supp}\backslash\{u_H\} \text{ and } F_S(u_H) = F_W(u_L) = 1,$$

i.e., F_S strictly first-order stochastically dominates F_W. Thus, at the upper bound of the subthreshold region, u_L, $\Pi_W(u_L) < \Pi_S(u_L)$. For $x \in (u_L, \theta)$, Π_S and Π_W are constant. At θ, the reward functions must jump up and each must meet their respective support lines in order for chasing the bonus to be a best response. The size of the jump will depend on the mass placed on θ. The larger the mass, the larger the jump from topping θ. Since Π_W is lower than Π_S for $x \in (u_L, \theta)$, and the component of the jump produced by the bonus is the same for both managers, the jump in W's reward function must be larger. Hence, S must place more mass on θ. However, both managers cannot place mass on θ, as this would result in a tie with positive probability (Remark 1). Thus, the mass placed on θ by W must equal 0 and the mass placed by S, which we denote by p_S^θ, must be positive. Lemma A.2 shows that, in any Eq2 equilibrium, $\alpha_W = 0$ and $\alpha_S > 0$, $\alpha_S = r\, F_W(0)$ and $\alpha_W = r\, F_S(0)$ imply that W, and only W, places some probability mass on 0. Except perhaps at 0 and θ, both equilibrium performance distributions are continuous (Remark 2.(c)). All points in the support of the managers' distributions lie on their support lines, which are affine. Thus, both managers, conditioned on choosing performance levels in $(0, u_L]$, where $u_L < \theta$ submit uniformly distributed performance; both managers, conditioned on choosing performance levels in $(\theta, u_H]$ submit uniformly distributed performance.

Let p_i^h, $i = S, W$, represent the probability that manager i targets the superthreshold region $(\theta, u_H]$, i.e., chooses performance levels in this region. The arguments above have shown that the performance distributions of the managers can be described as follows:

(a) manager S targets the superthreshold region $(\theta, u_H]$ with probability p_S^h and, conditioned on targeting this region, randomizes using a uniform distribution Unif$[\theta, u_H]$; S puts point mass on θ with probability p_S^θ and targets the subthreshold competition $(0, u_L]$ with probability $1 - p_S^h - p_S^\theta$ and, conditioned on targeting this region, randomizes using a Unif$[0, u_L]$ distribution;
(b) manager W targets the superthreshold region $(\theta, u_H]$ with probability p_W^h and, conditioned on targeting this region, randomizes using a uniform distribution Unif$[\theta, u_H]$; W puts point mass on 0 with probability p_W^0 and targets the subthreshold competition $(0, u_L]$ with probability $1 - p_W^h - p_W^0$ and, conditioned on targeting this region, randomizes using a Unif$[0, u_L]$ distribution.

For manager S, her reward at bonus threshold should meet her support line. As α_S is just the probability that manager W plays zero, we have

$$\alpha_S + \beta_S\,\theta = r\,p_W^0 + \beta_S\,\theta = r\,(1 - p_W^h) + 1. \tag{A-12}$$

Meanwhile the slope over $[0, u_L]$ region should be the same as the slope over $[\theta, u_H]$ region, which both equal to β_S/r, i.e.,

$$\frac{\beta_S}{r} = \frac{1 - p_W^h - p_W^0}{u_L} = \frac{p_W^h}{u_H - \theta}. \tag{A-13}$$

Similarly, for manager W, reward at θ meets his support line, i.e.,

$$\beta_W\, r = r\,(1 - p_S^h) + 1. \tag{A-14}$$

Meanwhile the slope over $[0, u_L]$ region should be the same as the slope over $[\theta, u_H]$ region, which both equal to β_W/r, i.e.,

$$\frac{\beta_W}{r} = \frac{1 - p_S^h - p_S^\theta}{u_L} = \frac{p_S^h}{u_H - \theta}. \tag{A-15}$$

Thus, we have

$$\frac{1 - p_W^0 - p_W^h}{1 - p_S^\theta - p_S^h} = \frac{p_W^h}{p_S^h},$$

which implies that

$$\frac{p_W^h}{p_S^h} = \frac{1 - p_W^0}{1 - p_S^\theta}.$$

Because both managers use up their capacities in equilibrium, we have

$$(1 - p_S^h - p_S^\theta)\,\frac{u_L}{2} + \theta\,p_S^\theta + p_S^h\,\frac{\theta + u_H}{2} = \mu_S, \tag{A-16}$$

$$(1 - p_W^h - p_W^0)\,\frac{u_L}{2} + p_W^h\,\frac{\theta + u_H}{2} = \mu_W. \tag{A-17}$$

Combining the six equations (A-12), (A-13), (A-14), (A-15), (A-16) and (A-17), we are able to solve for the six unknowns $\{u_L, u_H, p_S^h, p_W^h, p_S^\theta, p_W^0\}$. The solution can be expressed as follows,

$$p_S^\theta = \frac{\mu_S - \mu_W}{\theta + r\,\mu_W}, \qquad p_S^h = \frac{(1+r)\,(u_H - \theta)}{r\,u_H}, \tag{A-18}$$

$$p_W^0 = \frac{(1+r)\,(\mu_S - \mu_W)}{\theta + r\,\mu_S}, \qquad p_W^h = \frac{u_H - \theta}{r\,(\theta - u_L)}, \tag{A-19}$$

$$u_H = \frac{2\,(1+r)\,(\theta + r\,\mu_S)\,(\theta + r\,\mu_W)^2}{(\theta + r\,\mu_S)^2 + (1+r)^2\,(\theta + r\,\mu_W)^2}, \tag{A-20}$$

$$u_L = \theta - \frac{2\,(\theta + r\,\mu_S)^2\,(\theta + r\,\mu_W)}{(\theta + r\,\mu_S)^2 + (1+r)^2\,(\theta + r\,\mu_W)^2}. \tag{A-21}$$

Finally, from the results above it is not hard to calculate the contest reward functions for both managers. The reward function for manager S from submitting performance x is as follows,

$$\Pi_S(x) = \begin{cases} r\left(p_W^0 + (1 - p_W^0 - p_W^h)\,\dfrac{x}{u_L}\right) & \text{for } x \le u_L, \\ r\,(1 - p_W^h) & \text{for } x \in (u_L, \theta), \\ r\left((1 - p_W^h) + p_W^h\,\dfrac{x - \theta}{u_H - \theta}\right) + 1 & \text{for } x \in [\theta, u_H], \\ r + 1 & \text{for } x > u_H. \end{cases} \tag{A-22}$$

Correspondingly, the reward function for manager W is

$$\Pi_W(x) = \begin{cases} r\,(1 - p_S^\theta - p_S^h)\,\dfrac{x}{u_L} & \text{for } x \le u_L, \\ r\,(1 - p_S^\theta - p_S^h) & \text{for } x \in (u_L, \theta), \\ r\left((1 - p_S^h) + p_S^h\,\dfrac{x - \theta}{u_H - \theta}\right) + 1 & \text{for } x \in [\theta, u_H], \\ r + 1 & \text{for } x > u_H. \end{cases} \tag{A-23}$$

Moreover, the parameters of support lines satisfy

$$\alpha_S = r\,p_W^0, \qquad \beta_S = \frac{r\,(1 - p_W^h - p_W^0)}{u_L}, \qquad \beta_W = \frac{r\,(1 - p_S^h - p_S^\theta)}{u_L}.$$

Proof of Lemma 2 Suppose, to obtain a contradiction, that there exists an equilibrium in which only W chases the bonus. Thus, r is in the support of W's performance distribution. The reward from choosing $x = \theta$, $r + 1$, lies on W's support line, i.e.,

$$\alpha_W + \beta_W\,\theta = r + 1. \tag{A-24}$$

Similarly, because not chasing the bonus is a best response for S, the payoff to S from not chasing the bonus lies weakly below S's support line, i.e.,

$$\alpha_S + \beta_S\,\theta \ge r + 1. \tag{A-25}$$

Equations (A-24) and (A-25) imply that

$$\alpha_S + \beta_S\,\theta \ge \alpha_W + \beta_W\,\theta. \tag{A-26}$$

Now let $\bar{x} = \max(\text{Supp}_S)$. The capacity constraint implies that $\bar{x} > 0$. The hypothesis that S is not chasing the bonus implies that $\bar{x} < \theta$. For $x < \theta$, the reward to S for performance x, $\Pi_S(x) = r\,F_W(x)$, and, similarly, $\Pi_W(x) = r\,F_S(x)$. Moreover, over $[0, \theta)$, the supports of F_S and F_W coincide (Remarks 2.(a) and 2.(d)) and, hence, $\bar{x} \in \text{Supp}_S \cap \text{Supp}_W$. This implies that $\bar{x}$ lies on both S's and W's support lines. Because W is chasing the bonus and S is not, $F_W(\bar{x}) < 1 = F_S(\bar{x})$. These observations imply that

$$\alpha_S + \beta_S\,\bar{x} = r\,F_W(\bar{x}) < r = r\,F_S(\bar{x}) = \alpha_W + \beta_W\,\bar{x},$$

which, in turn, implies that

$$\alpha_S + \beta_S\,\bar{x} < \alpha_W + \beta_W\,\bar{x}, \quad \bar{x} < \theta. \tag{A-27}$$

Now note that, because S's capacity exceeds W's, it cannot be the case that the performance distribution of W first-order stochastically dominates the performance distribution of S. Because S is not chasing the bonus, this implies that there must exist at least one point, say x_o such that $x_o \in [0, \bar{x})$ and $F_W(x_o) > F_S(x_o)$. Using the same argument as used in the previous case, we see that

$$\alpha_S + \beta_S\,x_o = r\,F_W(x_o) > r\,F_S(x_o) = \alpha_W + \beta_W\,x_o,$$

which implies that,

$$\alpha_S + \beta_S\,x_o > \alpha_W + \beta_W\,x_o, \quad x_o \in [0, \bar{x}). \tag{A-28}$$

Inspection shows that equations (A-26), (A-27), and (A-28) cannot be simultaneously satisfied for any choice of multipliers, α_S, β_S, α_W, β_W, and thereby establishes the contradiction.

Proof of Lemma 3 First note that, in an Eq1 equilibrium, because, by hypothesis, S is chasing the bonus and W is not, if S submits performance θ, S wins both the rank and bonus reward. Performance $x > \theta$ will not produce a larger reward and requires more capacity. Thus, θ is in the support of S's performance distribution, F_S, and θ is not in the support of W's performance distribution, F_W, and $x > \theta$ is not in the support of either managers' distributions.

A proof analogous to the proof given for Eq2 shows that, in any Eq1 equilibrium, $\alpha_S > 0$ and $\alpha_W = 0$. Then by the same argument used in the proof of Lemma 1, W must place positive mass on 0. Let p_W^0 be the weight that W places on 0. Because a performance of θ is sufficient to capture both the rank and bonus rewards, capacity has a positive shadow price, and, by hypothesis, S chases the bonus, the only point in the support of S's performance distribution that is larger than the maximum performance of W is θ. Thus, S places probability mass on θ. Let p_S^θ represent this weight. Let $u = \max(\text{Supp}_W)$ represent the upper bound on the performance of W. All $x \neq \theta$ are either in the supports of both managers' distribution or in the supports of neither. Except perhaps at 0 and θ, both equilibrium performance distributions are continuous (Remark 2.(c)). Moreover, Supp_W is connected (Remark 2.(b)) and the support of S's distribution, is the same as W's support except at $x = \theta$ (Remark 2.(a)). Thus, the support of W's performance distribution equals $[0, u]$ and the support of S's performance distribution equals $[0, u] \cup \{\theta\}$. All points in the support of the managers' distributions lie on their support lines, which are affine. Thus, conditioned on choosing a performance level $x \in (0, u]$, both managers randomize uniformly.

Thus, we know that in the equilibrium, manager S puts point mass on θ with probability p_S^θ and plays a uniform distribution Unif$[0,u]$ with probability $1-p_S^\theta$; whereas manager W randomizes between 0 with probability p_W^0 and uniform Unif$[0,u]$ with probability $1-p_W^0$. Our next task is to relate these parameters to the managers' capacities and the structure of bonus compensation, defined by θ and r.

For manager S, her support line intersects $(\theta, r+1)$, i.e.,

$$\alpha_S + \beta_S\,\theta = r+1. \tag{A-29}$$

Meanwhile if S submits u, she wins the reward r with probability one, i.e.,

$$\alpha_S + \beta_S\,u = r. \tag{A-30}$$

Combining equation (A-29) and (A-30) we have $\beta_S = 1/(\theta-u)$, $\alpha_S = r - u/(\theta-u)$, showing that

$$p_W^0 = \frac{\alpha_S}{r} = 1 - \frac{u}{r\,(\theta-u)}. \tag{A-31}$$

For manager W, we require that

$$r\,(1-p_S^\theta)\,\frac{x}{u} = \beta_W\,x, \qquad x\in[0,u], \tag{A-32}$$

Hence, $\beta_W = r\,(1-p_S^\theta)/u$.

Because both managers use up their capacities in equilibrium, we have

$$(1-p_S^\theta)\,\frac{u}{2} + \theta\,p_S^\theta = \mu_S, \tag{A-33}$$

$$(1-p_W^0)\,\frac{u}{2} = \mu_W. \tag{A-34}$$

Combining three equations (A-31), (A-33) and (A-34), we are able to solve for the three unknowns $\{u, p_S^\theta, p_W^0\}$. Given non-negativity, there is only one solution:

$$u = \frac{2\,\theta\,\mu_W}{\mu_W + \sqrt{2\,\theta\,\mu_W/r + \mu_W^2}}, \qquad p_S^\theta = \frac{2\,\mu_S - u}{2\,\theta - u}, \qquad p_W^0 = 1 - \frac{2}{u}\,\mu_W.$$

Finally, from the results above it is not hard to calculate the contest reward functions for both managers. The reward function for manager S is given by

$$\Pi_S(x) = \begin{cases} r\left(p_W^0 + (1-p_W^0)\,\dfrac{x}{u}\right) & \text{for } x \le u, \\ r & \text{for } x\in(u,\theta), \\ r+1 & \text{for } x \ge \theta. \end{cases} \tag{A-35}$$

Correspondingly, the reward function for manager W is given by

$$\Pi_W(x) = \begin{cases} r\,(1-p_S^\theta)\,\dfrac{x}{u} & \text{for } x \le u, \\ r\,(1-p_S^\theta) & \text{for } x\in(u,\theta), \\ r+1 & \text{for } x \ge \theta. \end{cases} \tag{A-36}$$

Moreover, the parameters of support lines satisfy

$$\alpha_S = r - \frac{u}{\theta-u}, \qquad \beta_S = \frac{1}{\theta-u}, \qquad \beta_W = \frac{r\,(1-p_S^\theta)}{u}.$$

Proof of Lemma 4 Using the same arguments as used in the proofs of Lemmas 1 and 3, one can show that, in an Eq0 equilibrium, which, by definition is an equilibrium in which the upper bound on the performance of both managers is less than the bonus threshold, the common support of the managers' performance distribution is an interval of the form $[0, \bar{x}]$, where $\bar{x} \in (0, \theta)$. For the same reasons as offered in the proofs of Lemmas 1 and 3, it must be the case that $\alpha_S > 0$ and $\alpha_W = 0$. Thus, the fact that managers' reward functions meets the managers' support lines on the support of the managers' performance distributions implies that

$$\forall x \in [0, \bar{x}],\ \Pi_W(x) = \ell_W(x) = \beta_W\, x = r\, F_S(x), \tag{A-37}$$

$$\forall x \in [0, \bar{x}],\ \Pi_S(x) = \ell_S(x) = \alpha_S + \beta_S\, x = r\, F_W(x). \tag{A-38}$$

Equation (A-37) can only be satisfied if F_S is uniformly distributed over $[0, \bar{x}]$. Because the capacity of S is μ_S, the capacity constraint is binding, and F_S is uniformly distributed, $\bar{x} = 2\,\mu_S$. Equation (A-37) thus implies that $\beta_W = r/(2\,\mu_S)$.

As implied by (A-38), $\alpha_S = r\, F_W(0)$. Thus, equation (A-38) shows that

$$\beta_S\, x = r\,(F_W(x) - F_W(0)), \quad x \in [0, 2\,\mu_S]. \tag{A-39}$$

Define the distribution function G_W as follows:

$$G_W(x) = \frac{F_W(x) - F_W(0)}{1 - F_W(0)}, \quad x \in [0, 2\,\mu_S].$$

Note that equation (A-39), implies that, for $x \in [0, 2\,\mu_S]$, $G_W(x) = c\,x$ where c is some positive constant. The lower and upper bounds for the support of G_W are 0 and $2\,\mu_S$, i.e., G_W is uniformly distributed between 0 and $2\,\mu_S$. Thus, the mean of G_W equals μ_S. Finally, note that

$$F_W(x) = F_W(0) + (1 - F_W(0))\, G_W(x) = p_W^0 + (1 - p_W^0)\, G_W(x).$$

The mean performance of W must satisfy W's capacity constraint and thus, because the mean of G_W equals μ_S, it must be the case that

$$\mu_W = (1 - p_W^0)\,\mu_S,$$

which implies that $p_W^0 = (\mu_S - \mu_W)/\mu_S$.

Proof of Lemma 5

The proof of this lemma is tedious and so we have broken the steps into a series of results.

To initiate the proof, for fixed manager capacities, μ_S and μ_W, define the following functions:

$$\text{ConEq0}(\theta, r) = \left(1 - \frac{\mu_W}{\mu_S}\right) r + \frac{\mu_W}{\mu_S}\,\frac{\theta}{2\,\mu_S}\, r - (r+1), \tag{A-40}$$

$$\text{ConEq1}(\theta, r) = \frac{2\,\theta\,(\theta - \mu_S)}{(2\,\theta - u)\,u}\, r - (r+1),\ \text{where } u = \frac{2\,\theta\,\mu_W}{\mu_W + \sqrt{2\,\theta\,\mu_W/r + \mu_W^2}}, \tag{A-41}$$

$$\text{ConEq2}(\theta, r) = u_H - \theta,\ \text{where } u_H = \frac{2\,(1+r)\,(\theta + r\,\mu_S)\,(\theta + r\,\mu_W)^2}{(\theta + r\,\mu_S)^2 + (1+r)^2\,(\theta + r\,\mu_W)^2}. \tag{A-42}$$

Result A.2. For any bonus compensation package $(\theta, r) \in (\mu_S, \infty) \times (0, \infty)$,

(i) An Eq0 equilibrium exists only if $\text{ConEq0}(\theta, r) \geq 0$.
(ii) An Eq1 equilibrium exists only if $\text{ConEq1}(\theta, r) \geq 0$.
(iii) An Eq2 equilibrium exists only if $\text{ConEq2}(\theta, r) > 0$.

PROOF: This result is fairly obvious. In an Eq0 equilibrium, $\alpha_S = r\,(1 - \frac{\mu_W}{\mu_S})$, and $\beta_S = \frac{\mu_W}{\mu_S} \frac{r}{2\,\mu_S}$. Thus $\text{ConEq0}(\theta, r) \geq 0$ is equivalent to not chasing the bonus being a best response for S.

As shown in the proof of Lemma 3, in an Eq1 equilibrium, $\alpha_W = 0$ and $\beta_W = r\,(1 - p_S^\theta)/u$, where u and p_S^θ are defined in Lemma 3. Substituting the parameters shows that $\text{ConEq1} \geq 0$ is equivalent to the condition that $\beta_W \theta \geq (r+1)$, the condition for not chasing the bonus to be a best reply for W.

By definition, in an Eq2 equilibrium, both managers pursue rank competition at performance levels in excess of the reward threshold, θ. Thus, the expression for the upper bound of the superthreshold region, u_H, defined in equation (A-20), must exceed θ. $\square$

Result A.3. For any bonus compensation package $(\theta, r) \in (\mu_S, \infty) \times (0, \infty)$, $\text{ConEq0}(\theta, r) \geq 0$ if and only if an Eq0 equilibrium exists.

PROOF: As stated above, $\text{ConEq0}(\theta, r) \geq 0$ implies that not chasing the bonus is a best reply for S, because $\text{ConEq0}(\theta, r) \geq 0$ implies that $\theta > 2\,\mu_S$, the upper bound of performance in an Eq0 equilibrium. Thus, at $x = 2\,\mu_S$, the reward to both managers is r. Because the support lines cross at $x = 2\,\mu_S$ and because $\alpha_S > 0$ and $\alpha_W = 0$, for $x \geq 2\,\mu_S$, $\beta_W\, x \geq \alpha_S + \beta_S\, x$. Thus, $\alpha_S + \beta_S\,\theta \geq r + 1$ implies that $\beta_W\,\theta \geq r+1$. Because, $\text{ConEq0}(\theta, r) \geq 0$ is equivalent to $\alpha_S + \beta_S\,\theta \geq r+1$, $\text{ConEq0}(\theta, r) \geq 0$ also implies that not chasing the bonus is a best reply for W. $\square$

Result A.4. For any bonus compensation package $(\theta, r) \in (\mu_S, \infty) \times (0, \infty)$, an Eq1 equilibrium exists if and only if $\text{ConEq1}(\theta, r) \geq 0$ and $\text{ConEq0}(\theta, r) < 0$.

PROOF: For an Eq1 equilibrium to exist it must be the case that $p_W^0 \in (0,1)$, $p_S^\theta \in (0,1]$, and $u < \theta$, where p_W^0, p_S^θ, r and u are defined in Lemma 3. Using the definition of p_S^θ and the assumption that $\theta > \mu_S$, we see that $p_S^\theta < 1$. Inspecting the definition of p_W^0 provided in Lemma 3 shows that $p_W^0 < 1$. Using the definitions in Lemma 3, we see that

$$u = \frac{2\,\theta\,\mu_W}{\mu_W + \sqrt{2\,\theta\,\mu_W/r + \mu_W^2}} \text{ and } \frac{2\,\mu_W}{\mu_W + \sqrt{2\,\theta\,\mu_W/r + \mu_W^2}} < 1.$$

Hence, $u < \theta$. Inspection of the definition of u provided by Lemma 3 shows that $u > 0$.

Now consider p_W^0. Algebraic simplification shows that ConEq1 can be expressed at follows:

$$\text{ConEq1}(\theta, r) = \mathcal{S}(\theta, r)\, \frac{\theta + r\,\mu_W}{\theta\,\sqrt{\mu_W\,(2\,\theta/r + \mu_W)}}, \text{ where}$$

$$\mathcal{S}(\theta, r) = (\theta - \mu_S) - \frac{\theta + r\,\mu_S}{\theta + r\,\mu_W}\,\sqrt{\mu_W\ (2\,\theta/r + \mu_W)}. \tag{A-43}$$

Thus,

$$\text{sgn}\big(\text{ConEq1}(\theta, r)\big) = \text{sgn}\big(\mathcal{S}(\theta, r)\big). \tag{A-44}$$

Thus, if ConEq1 ≥ 0, then using the definition of u in Lemma 3, and equations (A-43) and (A-44), we see that

$$\mathcal{S}(\theta, r) \geq 0 \Rightarrow u \geq \frac{2\,\theta\,\mu_W}{\mu_W + (\theta - \mu_S)\ \frac{\theta + r\,\mu_W}{\theta + r\,\mu_S}}.$$

The right-hand side of this expression is increasing in r and

$$\lim_{r \to 0} \frac{2\,\theta\,\mu_W}{\mu_W + (\theta - \mu_S)\ \frac{\theta + r\,\mu_W}{\theta + r\,\mu_S}} = 2\,\mu_W\,\frac{\theta}{\theta - (\mu_S - \mu_W)} > 2\,\mu_W.$$

Thus, $u > 2\,\mu_W$. This implies, using the definition of p_W^0 provided in Lemma 3, that $p_W^0 > 0$.

Now consider p_S^θ. First note that because $u < \theta$, $p_S^\theta > 0$ if and only if $2\,\mu_S > u$. Using the quadratic formula, and the definition of u provided in Lemma 3 we see that

$$\text{ConEq0} < 0 \Longleftrightarrow \mu_S > \mu_S^o = \frac{\sqrt{2\,\theta\,\mu_W/r + \mu_W^2} - \mu_W}{2}\, r. \tag{A-45}$$

Next note, given the definition of u in Lemma 3,

$$2\,\mu_S - u = 2\,\mu_S - \frac{2\,\theta\,\mu_W}{\mu_W + \sqrt{2\,\theta\,\mu_W/r + \mu_W^2}}.$$

At $\mu_S = \mu_S^o$,

$$2\,\mu_S - \frac{2\,\theta\,\mu_W}{\mu_W + \sqrt{2\,\theta\,\mu_W/r + \mu_W^2}} = 0. \tag{A-46}$$

Because, the left hand side of equation (A-46) is increasing in μ_S, equations (A-45) and (A-46) imply that

$$p_S^\theta > 0 \Longleftrightarrow 2\,\mu_S - u > 0 \Longleftrightarrow \mu_S > \mu_S^o \Longleftrightarrow \text{ConEq0} < 0.$$

□

Result A.5. For any bonus compensation package $(\theta, r) \in (\mu_S, \infty) \times (0, \infty)$, an Eq2 equilibrium exists if and only if ConEq2$(\theta, r) > 0$.

Proof: An Eq2 equilibrium exists if and only if all the parameters specified in Lemma 1 satisfy their relevant range restrictions. First note that u_L, as defined in the Proof of Lemma 1 by equation (A-21), can be shown by algebraic manipulation to be equal to

$$\frac{\theta\ (\theta - \mu_S)\ (\theta\,r + 2\,\theta + r\,\mu_S) + 2\,r\ (\theta - \mu_S)\ (\theta\,r + 2\,\theta + r\,\mu_S)\ \mu_W + r\,(r+1)^2\,\theta\,\mu_W^2}{(1 + 2\,(1 + 1/r))\ \theta^2 + 2\,\theta\,\mu_S + r\,\mu_S^2 + 2\,(r+1)^2\,\theta\,\mu_W + r\,(r+1)^2\,\mu_W^2} > 0.$$

Thus, $u_L > 0$. Inspecting the definition of u_L provided by equation (A-21) shows that $u_L < \theta$. Now consider p_W^0. The definition of p_W^0 provided in Lemma 1 states that

$$p_W^0 = \frac{(r+1)\,(\mu_S - \mu_W)}{\theta + r\,\mu_S},$$

and

$$0 < \frac{(r+1)\,(\mu_S - \mu_W)}{\theta + r\,\mu_S} < \frac{(r+1)\,(\mu_S - \mu_W)}{\mu_S + r\,\mu_S} = \frac{\mu_S - \mu_W}{\mu_S} < 1.$$

So, $p_W^0 \in (0,1)$. Now consider p_S^θ. The definition of p_S^θ provided in Lemma 1 states that

$$p_S^\theta = \frac{\mu_S - \mu_W}{\theta + r\,\mu_W},$$

and

$$0 < \frac{\mu_S - \mu_W}{\theta + r\,\mu_W} < \frac{\mu_S - \mu_W}{\mu_S + r\,\mu_W} < 1.$$

Now suppose that $p_S^h \in (0,1)$. The definitions p_S^h and p_W^h provided by equations (A-18) and (A-19) imply that

$$p_W^h = \frac{\theta + r\,\mu_W}{\theta + r\,\mu_S}\,p_S^h.$$

Thus if $p_S^h \in (0,1)$ then $p_W^h \in (0,1)$. So an Eq2 equilibrium will exist if $u_H > \theta$ and $p_S^h \in (0,1)$.

First we show that it is always the case that $p_S^h < 1$. Note that the definition of u_H provided by equation (A-20) implies that $p_S^h \geq 1$ if and only if

$$\frac{2\,(\theta + r\,\mu_S)\,(\theta + r\,\mu_W)^2}{(\theta + r\,\mu_S)^2 + (r+1)^2\,(\theta + r\,\mu_W)^2} \geq \theta. \tag{A-47}$$

Condition (A-47) can only be satisfied if

$$\begin{aligned} 2\,(\theta + r\,\mu_S)\,(\theta + r\,\mu_W)^2 - \left((\theta + r\,\mu_S)^2 + (r+1)^2\,(\theta + r\,\mu_W)^2\right)\theta = \\ -\,\theta\,\mu_W^2\,r^4 - 2\,r\,\theta^2\,(\theta - \mu_W) - 2\,r^3\,\mu_W\,\left(\theta^2 + \mu_W\,(\theta - \mu_S)\right) - \\ r^2\,\theta\,\left(\theta^2 + \left(\mu_S^2 - \mu_W^2\right) + 4\,\mu_W\,(\theta - \mu_S)\right) > 0, \end{aligned}$$

which is clearly impossible. Thus, $p_S^h < 1$, which implies that $p_W^h < 1$.

Finally note that from the definition of p_W^h provided by equation (A-18), $p_W^h > 0$ if and only if $u_H > \theta$. Thus, an Eq1 equilibrium exists if and only if $u_H > \theta$, i.e., ConEq2 > 0. $\square$

Result A.6. $\operatorname{sgn}\big(\text{ConEq2}\big) = -\operatorname{sgn}\big(\text{ConEq1}\big)$.

Proof: Let

$$\mathcal{S}_2(\theta, r) = (\theta - \mu_S)^2 - \mu_W\,(2\,\theta/r + \mu_W)\left(\frac{\theta + r\,\mu_S}{\theta + r\,\mu_W}\right)^2.$$

Algebraic simplification shows that

$$\mathrm{ConEq2}(\theta, r) = -K\,\mathcal{S}_2(\theta, r), \text{ where}$$

$$K = \frac{r^2\ (\theta + r\,\mu_W)^{\,2}}{\theta\ \Big(((r+1)^2+1)\ (\theta + r\,\mu_W)^2 + (\mu_S - \mu_W)\ r\ \big((\theta + r\,\mu_W) + (\theta + r\,\mu_S)\big)\Big)}.$$

$K > 0$, so

$$\operatorname{sgn}\big(\mathrm{ConEq2}\big) = -\operatorname{sgn}\big(\mathcal{S}_2(\theta, r)\big). \tag{A-48}$$

Recall $\mathcal{S}$, defined in equation (A-43), and note that

$$\operatorname{sgn}\big(\mathcal{S}_2(\theta, r)\big) = \operatorname{sgn}\big(\mathcal{S}(\theta, r)\big). \tag{A-49}$$

The result follows from (A-48), (A-49), and (A-44). □

Result A.7. Define the parametric curves, with parameter $y > 0$ as follows:

$$\Theta_0^1(y) = \frac{2\,y\,\mu_S}{\sqrt{2\,y\,\mu_W + \mu_W^2} - \mu_W}, \qquad \mathcal{R}_0^1(y) = \frac{2\,\mu_S}{\sqrt{2\,y\,\mu_W + \mu_W^2} - \mu_W};$$

$$\Theta_1^2(y) = \mu_S + \frac{y + \mu_S}{y + \mu_W}\sqrt{2\,y\,\mu_W + \mu_W^2}, \quad \mathcal{R}_1^2(y) = \frac{1}{y}\left(\mu_S + \frac{y + \mu_S}{y + \mu_W}\sqrt{2\,y\,\mu_W + \mu_W^2}\right).$$

These parametric curves have the following properties:

(i) $\mathcal{R}_0^1(y)$ and $\mathcal{R}_1^2(y)$ are strictly decreasing and map $(0, \infty)$ onto $(0, \infty)$.
(ii) $\Theta_0^1(y)/\mathcal{R}_0^1(y) = \Theta_1^2(y)/\mathcal{R}_1^2(y) = y$.
(iii) $\mathcal{R}_1^2(y) < \mathcal{R}_0^1(y)$.
(iv) $\mathrm{ConEq2}(\theta, r) = 0 \iff (\theta, r) = (\Theta_1^2(y), \mathcal{R}_1^2(y))$ for some $y > 0$.
(v) $\mathrm{ConEq1}(\theta, r) = 0 \iff (\theta, r) = (\Theta_1^2(y), \mathcal{R}_1^2(y))$ for some $y > 0$.
(vi) $\mathrm{ConEq0}(\theta, r) = 0 \iff (\theta, r) = (\Theta_0^1(y), \mathcal{R}_0^1(y))$ for some $y > 0$.

Proof: Property (i) follows from differentiation. Property (ii) follows from inspection. To establish property (iii), note that

$$\mathcal{R}_1^2(y) - \mathcal{R}_0^1(y) = -(\mu_S - \mu_W)\,\frac{y\,\sqrt{\mu_W\,(2\,y + \mu_W)}}{\mu_W\,(y + \mu_W)} < 0.$$

To establish parts (iv), (v), first note that, by equations (A-44), (A-48), and (A-49), parts (iv), (v) are equivalent to the assertion that

A1: $\quad \mathcal{S}(\theta, r) = 0 \iff$ there exists $y > 0$ such that $(\theta, r) = (\Theta_1^2(y), \mathcal{R}_1^2(y))$.

Inspecting the definition of $\mathcal{S}$ provided by equation (A-43), we see that $\mathcal{S}(\theta, r) = 0$ is equivalent to the assertion that

A2: $\quad$ there exists $y > 0$ such that $\theta = \mu_S + \dfrac{y + \mu_S}{y + \mu_W}\sqrt{\mu_W\ (2\,y + \mu_W)}$ and $y = \theta/r$.

A1 and A2 are clearly equivalent.

Property (vi) follows from substitution of the definitions of Θ_0^1 and $\mathcal{R}_0^1$, provided by Result A.7, into ConEq0 (equation (A-40)). □

Result A.8. If $(\theta_1, r) = (\Theta_0^1(y_1), \mathcal{R}_0^1(y_1))$ and $(\theta_2, r) = (\Theta_1^2(y_2), \mathcal{R}_1^2(y_2))$, then $\theta_1 > \theta_2$.

PROOF: By parts (i) and (iii) of Result A.7, if $r = \mathcal{R}_0^1(y_1) = \mathcal{R}_1^2(y_2)$, then $y_1 > y_2$. By part (ii) of Result A.7, $\theta_1/r = y_1$ and $\theta_2/r = y_2$. Thus $\theta_1 > \theta_2$. □

Result A.9. For any fixed $r > 0$,

(i) $\lim_{\theta\to\mu} \text{ConEq0}(\theta, r) < 0$ and $\lim_{\theta\to\infty} \text{ConEq0}(\theta, r) > 0$.
(ii) $\lim_{\theta\to\mu} \text{ConEq1}(\theta, r) < 0$ and $\lim_{\theta\to\infty} \text{ConEq1}(\theta, r) > 0$.
(iii) $\lim_{\theta\to\mu} \text{ConEq2}(\theta, r) > 0$ and $\lim_{\theta\to\infty} \text{ConEq2}(\theta, r) < 0$.

PROOF: These assertions follow from straightforward calculations. □

Result A.10. For each $r > 0$,

(i) there exits a unique $\theta_2 > \mu_S$, such that $\text{ConEq1}(\theta_2, r) = \text{ConEq2}(\theta_2, r) = 0$; if $\theta < \theta_2$, $\text{ConEq1}(\theta_2, r) < 0$ and $\text{ConEq2}(\theta_2, r) > 0$; if $\theta > \theta_2$, $\text{ConEq1}(\theta_2, r) > 0$ and $\text{ConEq2}(\theta_2, r) < 0$.
(ii) There exists a unique $\theta_1 > \mu_S$, such that $\text{ConEq0}(\theta_1, r) = 0$; if $\theta < \theta_1$, $\text{ConEq0}(\theta_1, r) < 0$; if $\theta > \theta_1$, $\text{ConEq0}(\theta_1, r) > 0$.
(iii) $\theta_1 > \theta_2$.

PROOF: The proofs of parts (i) and (ii) are identical. So we will only present the proof of (i). Suppose that $\text{ConEq1}(\theta_2', r) = 0$ and $\text{ConEq1}(\theta_2'', r) = 0$. Then, by parts (v) and (iv) of Result A.7, $(\theta_2', r) = (\Theta_1^2(y'), \mathcal{R}_1^2(y'))$ for some $y' > 0$ and $(\theta_2'', r) = (\Theta_1^2(y''), \mathcal{R}_1^2(y''))$ for some $y'' > 0$. By part (i) of Result A.7, $\mathcal{R}_1^2(y') = \mathcal{R}_1^2(y'')$ implies that $y' = y''$ and thus $\theta_2' = \theta_2''$.

The fact that there is a unique $\theta = \theta_2$, such that $\text{ConEq1}(\theta_2, r) = 0$, combined with the continuity of ConEq1 implies that the sign of ConEq1 is constant for $\theta < \theta_2$ and is constant for $\theta > \theta_2$. Part (i) thus follows from part (ii) of Result A.10 and Result A.6.

Part (iii) follows from parts (v), (iv), (vi) of Result A.7, and Result A.8. □

Result A.11. Define

$$
\begin{aligned}
E &= (\mu_S, \infty) \times (0, \infty), \\
E_{01} &= \{(\theta, r) : \exists y > 0 \text{ such that } \theta \geq \Theta_0^1(y) \text{ and } r = \mathcal{R}_0^1(y)\}. \\
E_{12} &= \{(\theta, r) : \exists y > 0 \text{ such that } \theta \geq \Theta_1^2(y) \text{ and } r = \mathcal{R}_1^2(y)\}.
\end{aligned}
$$

Then

(i) $(\theta, r) \in E_{01} \Longleftrightarrow \text{ConEq0}(\theta, r) \geq 0$,
(ii) $(\theta, r) \in E_{12} \Longleftrightarrow \text{ConEq1}(\theta, r) \geq 0$,

(iii) $(\theta, r) \in E \backslash E_{12} \Longleftrightarrow \text{ConEq2}(\theta, r) > 0$.

PROOF: The proofs of parts (i), (ii) are virtually identical. Thus we will only prove part (i). We first prove sufficiency. Note that if $(\theta, r) = (\Theta_0^1(y), \mathcal{R}_0^1(y))$, then by part (vi) of Result A.7, $\text{ConEq0}(\theta, r) = 0$. Next suppose that $r = \mathcal{R}_0^1(y)$ and $\theta > \Theta_0^1(y)$. Let $\theta_1 = \Theta_0^1(y)$. Part (ii) of Result A.10 shows that θ_1 is the unique θ satisfying $\text{ConEq0}(\theta, r) = 0$, and implies, because $\theta > \theta_1$, $\text{ConEq0}(\theta, r) > 0$.

To prove necessity, suppose that $(\theta, r) \notin E_{01}$. Let y' be the unique $y > 0$ such that $\mathcal{R}_0^1(y) = r$ (the existence and uniqueness of y' follows from property (i) in Result A.7). Thus, $\mathcal{R}_0^1(y') = r$. If $(\theta, r) \notin E_{01}$, then it must be the case that $\theta < \Theta_0^1(y)$, using the same argument as used above for sufficiency, we see that Part (ii) of Result A.10 implies that $\text{ConEq0}(\theta, r) < 0$.

Part (iii) follows from part (ii) and Result A.6. □

Proof of Lemma 5 Result A.11, and Results A.3, A.4, and A.5 show that an Eq0 equilibrium can be sustained if and only if $(\theta, r) \in E_{01}$; an Eq1 equilibrium can be sustained if and only if $(\theta, r) \in E_{12} \backslash E_{01}$; an Eq2 equilibrium can be sustained if and only if $(\theta, r) \in E \backslash E_{12}$. We need only show that these three sets are disjoint. Clearly E_{01} and $E_{12} \backslash E_{01}$ are disjoint as are $E_{12} \backslash E_{01}$ and $E \backslash E_{12}$. Now consider E_{01} and $E \backslash E_{12}$. Result A.10 implies that $E_{01} \subset E_{12}$. Thus E_{01} and $E \backslash E_{12}$ are disjoint.

Result A.12. For $\theta_o > \mu_S$, $\theta_o > \inf_{y>0} \Theta_1^2(y)$ if and only if for some $r > 0$ there exists $\theta < \theta_o$ such that (θ, r) sustains an Eq1 equilibrium.

PROOF: We start by proving sufficiency. Suppose that $\theta_o > \inf_{y>0} \Theta_1^2(y)$, then there exists some $y > 0$, say y_2, such that $\theta_o > \Theta_1^2(y_2)$. Let $(\theta_2, r_o) = (\Theta_1^2(y_2), \mathcal{R}_1^2(y_2))$. By property (v) in Lemma A.7, $\text{ConEq1}(\theta_2, r_o) = 0$. The fact that $\theta_o > \theta_2$, implies, by part (i) of Result A.10, that $\text{ConEq1}(\theta_o, r_o) > 0$.

Parts (ii) and (iii) of Result A.10 show that there is a unique bonus threshold, θ_1, such that $\text{ConEq0}(\theta_1, r_o) = 0$ and that $\theta_1 > \theta_2$. Let $\theta^* = \min[\theta_1, \theta_o]$. Because $\theta_1 > \theta_2$ and $\theta_o > \theta_2$, $\theta^* > \theta_2$. By definition, $\theta^* \leq \theta_o$. Hence, if $\theta \in (\theta_2, \theta^*)$, $\theta < \theta_o$, $\theta > \theta_1$ and $\theta < \theta_2$. Parts (ii) and (i) of Result A.10 and Result A.4 thus establish that (θ, r) sustains an Eq1 equilibrium.

To prove necessity, note simply that $\theta_o \leq \inf_{y>0} \Theta_1^2(y)$ implies that the set, $\{(\theta, r) : \theta < \theta^o\} \cap E_{12}$ is empty. And thus, by part (ii) of Result A.11, for all $r > 0$, if $\theta < \theta^o$ then $\text{ConEq1}(\theta_2, r_o) < 0$, and thus, by Result A.4, an Eq1 equilibrium cannot be sustained. □

Proof of Proposition 2 We first prove part (i): the ruin risk taking in the rank competition is less than in the bonus competition. Next to verify parts (ii) and (iii), we show that in Eq1 and Eq2 configurations ruin risk taking is less than in Eq0 configurations. Because manager performance distributions in Eq0 configuration are the same as their performance strategies under rank rewards, this establishes that ruin risk taking under mixed rewards is less than ruin risk taking under rank rewards. Given part (i) this implies that ruin risk taking under mixed rewards is less than ruin risk taking under bonus and rank rewards.

(i) This follows from straightforward calculations. Using Result 1 and Lemma 4 we see that the difference between ruin risk taking under bonus and rank rewards is given by

$$\left(1 - \frac{\mu_W}{\theta}\right) - \left(1 - \frac{\mu_W}{\mu_S}\right) = \left(\frac{1}{\mu_S} - \frac{1}{\theta}\right) > 0.$$

(ii) First consider Eq1. Let $p_W^0(\text{Eq1})$ represent the probability that the weak manager places point mass on zero in an Eq1 equilibrium. Let $p_W^0(\text{Eq0})$ represent the probability that the weak manager places point mass on zero in an Eq0 equilibrium. As shown in Lemma 4, in Eq0,

$$p_W^0(\text{Eq0}) = 1 - \frac{\mu_W}{\mu_S}.$$

As shown in Lemma 3, in Eq1,

$$p_W^0(\text{Eq1}) = 1 - \frac{2}{u}\,\mu_W.$$

Lemma 3 shows, $u < 2\,\mu_S$, and thus $p_W^0(\text{Eq1}) < p_W^0(\text{Eq0})$. Thus, by result part (i), we know that ruin risk is less in Eq1 equilibria under mixed rewards is lower than ruin risk taking under rank or bonus rewards.

(iii) Now consider Eq2. Let $p_W^0(\text{Eq2})$ represent the probability that the weak manager places point mass on zero in an Eq2 equilibrium. Substituting the definitions of u_L, u_H and p_W^0 provided in Lemma 1 and equations (A-21) and (A-20) in this appendix shows that

$$p_W^0(\text{Eq2}) = p_W^0(\text{Eq0})\,\left(1 - \frac{\theta - \mu_S}{\theta + r\,\mu_S}\right). \tag{A-50}$$

It is always the case that

$$0 < \frac{\theta - \mu_S}{\theta + r\,\mu_S} < 1.$$

Thus, $p_W^0(\text{Eq2}) < p_W^0(\text{Eq0})$. Thus, by result part (i), we know that ruin risk is less in Eq2 equilibria under mixed rewards is lower than ruin risk taking under rank or bonus rewards.

Proof of Lemma 6

Proof of part (i).

Using the definition of p_W^0 in the Eq1 configuration provided in Lemma 3, we see that the probability of ruin risk taking, p_W^0 in this configuration is given by

$$p_W^0(\theta) = 1 - \left(\frac{\mu_W}{\theta} + \sqrt{\frac{\mu_W}{\theta}\left(\frac{2}{r} + \frac{\mu_W}{\theta}\right)}\right).$$

This expression is evidently increasing in θ.

Proof of part (ii).

Inspection of the definition of p_W^0 provided in Lemma 1 shows that p_W^0 is decreasing in θ.

Proof of Proposition 3 First, note that, as was shown in the proof of Result A.4, in all Eq1 equilibria, $p_W^0 > 0$. Thus, all such equilibria feature some ruin risk taking. As was shown in the proof of Result A.5, in all Eq2 equilibria, $p_W^0 > 0$. Thus, all such equilibria feature some ruin risk taking. Lemma 4 shows that ruin risk taking occurs in Eq0 equilibria. Thus, ruin risk taking is never 0 in any equilibrium configuration.

In order to prove that a sequence of bonus thresholds and bonus payments (θ_n, r_n) exist that support equilibria in which p_W^0 is arbitrarily small, we will employ the parametric curve, $y \hookrightarrow (\Theta_1^2(y), \mathcal{R}_1^2(y))$, defined in Result A.7.

Define the sequence of bonus packages (θ_n, r_n) as follows:

$$\theta_n = \Theta_1^2(n) = \mu_S + \frac{n+\mu_S}{n+\mu_W}\sqrt{2\,n\,\mu_W + \mu_W^2}, \tag{A-51}$$

$$r_n = \mathcal{R}_1^2(n) = \frac{1}{n}\left(\mu_S + \frac{n+\mu_S}{n+\mu_W}\sqrt{2\,n\,\mu_W + \mu_W^2}\right). \tag{A-52}$$

Result A.7 shows that for all n, (θ_n, r_n) sustains an Eq1 equilibrium and only an Eq1 equilibrium. Using the definition of u, the upper bond of the rank plus bonus competition in Eq1 configurations provided in Lemma 3, we see that

$$u_n = 2\,\mu_W\left(\frac{\mu_S}{\mu_W + \sqrt{\mu_W\,(2\,n+\mu_W)}} + \frac{n+\mu_S}{n+\mu_W}\,\frac{\sqrt{\mu_W\,(2\,n+\mu_W)}}{\mu_W + \sqrt{\mu_W\,(2\,n+\mu_W)}}\right). \tag{A-53}$$

The first term in the parentheses on the right-hand side of equation (A-53) converges to 0 as $n \to \infty$. The second term in the parentheses on the right-hand side of equation (A-53) converges to 1. Thus, $u_n \to 2\,\mu_W$. The definition of ruin risk taking, p_W^0 in Eq1 configurations (see Lemma 3) is $p_W^0 = 1 - 2\,\mu_W/u$. Because, $u_n \to 2\,\mu_W$, $1 - 2\,\mu_W/u_n \to 0$.

Using the same arguments, it can also be shown that $p_S^\theta(n) \to 0$ as $n \to \infty$. Thus, both F_S^n and $F_W^n \to \mathrm{Unif}[0, 2\,\mu_W]$ in distribution as $n \to \infty$.

Proof of Proposition 4

Proof of part (i).

We only need to show the case for Eq2 equilibria. First note that the upper bound of the superthreshold region for both the strong and the weak managers in the Eq2 configuration, u_H, provided by equation (A-20) in the proof of Lemma 1, is given by

$$u_H(r) = \frac{2\,(1+r)\,(\theta + r\,\mu_S)\,(\theta + r\,\mu_W)^2}{(\theta + r\,\mu_S)^2 + (1+r)^2\,(\theta + r\,\mu_W)^2}. \tag{A-54}$$

Next note that

$$\lim_{r\to 0} u_H(r) = \theta,$$

and

$$\lim_{r\to 0} u_H'(r) = \mu_W > 0,$$

Thus u_H is increasing for r sufficiently small. Thus $u_H > \theta$.

Proof of part (ii).

By definition provided in Result A.7, $\Theta_1^2(y) = 2\,\mu_S$, is equivalent to

$$2\,\mu_S = \mu_S + \frac{y+\mu_S}{y+\mu_W}\sqrt{2\,y\,\mu_W + \mu_W^2}.$$

The equation $\frac{y+\mu_S}{y+\mu_W}\sqrt{2\,y\,\mu_W + \mu_W^2} - \mu_S = 0$ has solution with $y > 0$ if and only if $y \hookrightarrow (\frac{y+\mu_S}{y+\mu_W})^2\,(2\,y\,\mu_W + \mu_W^2) - \mu_S^2$ has a positive real root. After algebraic simplification we see that

$$\begin{aligned} &\left(\frac{y+\mu_S}{y+\mu_W}\right)^2 (2\,y\,\mu_W + \mu_W^2) - \mu_S^2 = \frac{y}{(y+\mu_W)^2}\,\mathrm{Poly}(y), \quad \text{where} \\ &\mathrm{Poly}(y) = 2\,\mu_W\,y^2 + \mu_W^2\,\left(\left(\sqrt{5}-2\right) + \frac{\mu_S}{\mu_W}\right)\left(\left(2+\sqrt{5}\right) - \frac{\mu_S}{\mu_W}\right)\,y + 2\,\mu_S\,\mu_W^2. \end{aligned} \tag{A-55}$$

Taking the discriminant of Poly we obtain

$$\mathrm{Disc} = (\mu_S - \mu_W)^2\,\mu_W\,\mu_S\,\left(\frac{\mu_S}{\mu_W} + \frac{\mu_W}{\mu_S} - 6\right). \tag{A-56}$$

Let,

$$a = \frac{\mu_S}{\mu_W} > 1. \tag{A-57}$$

Disc ≤ 0 is equivalent to $a + 1/a - 6 \leq 0$. Thus, if $a \leq 3 + 2\sqrt{2}$, Disc ≤ 0 and thus Poly(y) either has no roots or a double root at 0. Hence, $\Theta_1^2(y) - 2\,\mu_S \geq 0$, for all $y > 0$. Thus, $\inf_{y>0}\Theta_1^2(y) \geq 2\,\mu_S$. Thus, (a) and (b) in part (ii) follows from Result A.12 that, for $\theta < 2\,\mu_S$, an Eq1 equilibrium cannot be sustained.

Next, we prove part (c). We have already shown that in the neighbor of 0, u_H is increasing in r. To verify that u_H is asymptotically decreasing, define the function

$$u_H^o(r) = u_H(1/r), \quad r > 0. \tag{A-58}$$

u_H will be asymptotically decreasing if and only if u_H^o is increasing in a neighborhood of 0. Taking the limit of $u_H^{o\,\prime}$ as $r \to 0$ yields

$$\lim_{r\to 0} u_H^{o\,\prime}(r) = 2\,(\theta - \mu_S) > 0. \tag{A-59}$$

Thus u_H^o is increasing in a neighborhood of 0 and thus u_H is asymptotically decreasing.

Last, we present the proof of part (d). The definition of u_H provided by Lemma 1 shows that

$$u_H - 2\,\mu_S = \frac{2\,\mu_S\,(\theta + r\,\mu_W)^2}{(1+r)^2\,(\theta + r\,\mu_W)^2 + (\theta + r\,\mu_S)^2}\,\chi(r), \text{ where} \tag{A-60}$$

$$\chi(r) = \frac{\theta - \mu_S}{\mu_S}\,(1+r) - \left(1 + \frac{r\,(\mu_S - \mu_W)}{\theta + r\,\mu_W}\right)^2, \quad r > 0. \tag{A-61}$$

χ and $u_H(r) - 2\,\mu_S$ have the same sign. χ is continuous and part ii.c show that, for r sufficiently small, $u_H(r) < 2\,\mu_S$ and, for r sufficiently large $u_H(r) > 2\,\mu_S$. Thus, there exist an $r > 0$ such that $\chi(r) = 0$. Suppose to obtain a contradiction that there exist, r_1, r_2, and r_3 such that $\chi(r_1) < 0$, $\chi(r_2) > 0$ and $\chi(r_3) < 0$. Then, because for r sufficiently large, $\chi(r) > 0$, χ would have at less three real roots. However, by the hypothesis that $\mu_S < 3\,\mu_W$,

$$\chi''(r) = \frac{2\,\theta\,(\mu_S - \mu_W)\,(\theta\,(3\,\mu_W - \mu_S) + 2\,r\,\mu_S\,\mu_W)}{(\theta + r\,\mu_W)^4} \geq 0,$$

Thus, χ is convex, and, for this reason, χ cannot have three roots.

Proof of part (iii).

From equations (A-55) and (A-57), we see that, if $a > 3 + 2\sqrt{2}$, then $\mathrm{Poly}(y)$ has two real roots. Because Poly is convex and is positive at $y = 0$, both roots are either positive or negative. Because $3+2\sqrt{2} > 2+\sqrt{5}$, $a > 3+2\sqrt{2}$ implies that $\left(2 + \sqrt{5}\right) - a < 0$. This implies that $\mathrm{Poly}'(0) < 0$ which, in turn, implies that both roots are positive. Hence, there exists a y-interval over which $\mathrm{Poly}(y) < 0$. This implies that, over this interval, $\Theta_1^2(y) - 2\,\mu_S < 0$. Hence, $\inf_{y>0}\Theta_1^2(y) < 2\,\mu_S$. Hence, part (iii) follows from Result A.12 that, there exists a feasible Eq1 equilibrium, in which the weak manager simply ignores the bonus chasing.

Proof of Proposition 5

Proof of part (i).

Because $\mathbb{E}[X_i] = \mathbb{E}[Y_i]$, $i = S, W$, X_i and Y_i can be ordered by FSD if and only if they are identically distributed (Theorem 1.A.8, Shaked and Shanthikumar, 2007). Since Lemma 1 and 3 show that they are not identically distributed, they are not ordered by FSD.

Proof of part (ii).

By Proposition 2, $F_i(0) < G_i(0)$, thus for some neighborhood of 0, $F_i(x) < G_i(x)$. If $x \geq \theta$, $F_i(x) = G_i(x) = 1$. Because F_i and G_i have the same mean, it is not possible for $F_i(x) \leq G_i(x)$ for all $x \in [0, \theta]$ and $F_i(x) < G_i(x)$ on a neighborhood of 0. Thus, F_i must cross G_i at some $x \in (0, \theta)$. Because G_i is flat over $(0, \theta)$ there can be at most one crossing. Thus, F_i crosses G_i once from below and because F_i and G_i have the same mean, this implies SSD (Theorems 3.A.44 and 4.A.35, Shaked and Shanthikumar, 2007).

Proof of part (iii).

Part (a). Note that mean performance equals capacity under both mixed and bonus rewards, thus the variance of performance will be smaller (larger) under mixed rewards if and only if $\mathbb{E}[X_i^2]$ is smaller (larger) than $\mathbb{E}[Y_i^2]$, $i = S, W$. Using the characterization of performance under bonus rewards (Result 1) and Eq2 equilibria under mixed rewards (Lemma 1) we see that, for $i = W$,

$$\begin{aligned}
\mathbb{E}[X_W^2] &= \frac{1 - p_W^h - p_W^0}{u_L} \int_0^{u_L} x^2\,\mathrm{d}x + \frac{p_W^h}{u_H - \theta} \int_\theta^{u_H} x^2\,\mathrm{d}x, \\
\mathbb{E}[Y_W^2] &= \theta^2 \left(\frac{\mu_W}{\theta}\right) = \theta\,\mu_W, \\
\mathbb{E}[Y_W^2] - \mathbb{E}[X_W^2] &= \theta\,\mu_W - \left(\frac{1 - p_W^h - p_W^0}{u_L} \int_0^{u_L} x^2\,\mathrm{d}x + \frac{p_W^h}{u_H - \theta} \int_\theta^{u_H} x^2\,\mathrm{d}x\right). \qquad \text{(A-62)}
\end{aligned}$$

Similarly, for $i = S$,

$$
\begin{aligned}
\mathbb{E}[X_S^2] &= \theta^2\, p_S^\theta + \left(\frac{1 - p_S^h - p_S^\theta}{u_L}\right) \int_0^{u_L} x^2\,\mathrm{d}x + \left(\frac{p_S^h}{u_H - \theta}\right) \int_\theta^{u_H} x^2\,\mathrm{d}x,\\
\mathbb{E}[Y_S^2] &= \theta^2 \left(\frac{\mu_S}{\theta}\right) = \theta\, \mu_S,\\
\mathbb{E}[Y_S^2] - \mathbb{E}[X_S^2] &= \mu_S\,\theta - \theta^2\, p_S^\theta - \left(\frac{1 - p_S^h - p_S^\theta}{u_L} \int_0^{u_L} x^2\,\mathrm{d}x + \frac{p_S^h}{u_H - \theta} \int_\theta^{u_H} x^2\,\mathrm{d}x\right).
\end{aligned} \tag{A-63}
$$

Again, using the definition of Eq2 equilibrium performance distributions in Lemma 1, we see that

$$
\mu_S\,\theta - \theta^2\, p_S^\theta = \left(\frac{\theta + r\,\mu_S}{\theta + r\,\mu_S}\right)(\theta\,\mu_W). \tag{A-64}
$$

Substituting in the definitions of p_S^h, p_W^h, p_S^θ, and p_W^0 in Lemma 1, shows that, in an Eq2 equilibrium

$$
\frac{1 - p_S^\theta - p_S^h}{1 - p_W^0 - p_W^h} = \frac{p_S^h}{p_W^h} = \frac{\theta + r\,\mu_S}{\theta + r\,\mu_W} > 1. \tag{A-65}
$$

Equation (A-65) implies that

$$
\begin{aligned}
\frac{1 - p_S^h - p_S^\theta}{u_L} &= \left(\frac{1 - p_S^h - p_S^\theta}{1 - p_W^h - p_W^0}\right) \frac{1 - p_W^h - p_W^0}{u_L} = \left(\frac{\theta + r\,\mu_S}{\theta + r\,\mu_W}\right) \frac{1 - p_W^h - p_W^0}{u_L},\\
\frac{p_S^h}{u_H - \theta} &= \left(\frac{p_S^h}{p_W^h}\right) \frac{p_W^h}{u_H - \theta} = \left(\frac{\theta + r\,\mu_S}{\theta + r\,\mu_W}\right) \frac{p_W^h}{u_H - \theta}.
\end{aligned} \tag{A-66}
$$

Substituting equations (A-64) and (A-66) into equation (A-63), and comparing the resulting expression with equation (A-62), shows that

$$
\begin{aligned}
\mathbb{E}[Y_S^2] - \mathbb{E}[X_S^2] &= \left(\frac{\theta + r\,\mu_S}{\theta + r\,\mu_W}\right) \left(\theta\,\mu_W - \Big(\frac{1 - p_W^h - p_W^0}{u_L} \int_0^{u_L} x^2\,\mathrm{d}x + \frac{p_W^h}{u_H - \theta} \int_\theta^{u_H} x^2\,\mathrm{d}x\Big)\right)\\
&= \left(\frac{\theta + r\,\mu_S}{\theta + r\,\mu_W}\right) \big(\mathbb{E}[Y_W^2] - \mathbb{E}[X_W^2]\big).
\end{aligned} \tag{A-67}
$$

Part (b). Let

$$
\Phi_i(x) = \int_0^x \big(F_i(t) - G_i(t)\big)\,\mathrm{d}t, \quad x \in [0, u_H],\ i = S, W.
$$

Because $\mathbb{P}[X_i \in [0, u_H]] = \mathbb{P}[Y_i \in [0, u_H]] = 1$ and the expectations of X_i and Y_i are equal, $\Phi_i(u_H) = 0$. Because for $t \in (\theta, u_H)$, $F_i(t) - G_i(t) < 0$, $\Phi_i(x)$ is strictly decreasing in t for $t \in (\theta, u_H)$. Because, Φ_i is continuous (Fundamental Theorem of Calculus), Φ_i being equal to 0 at u_H and strictly decreasing over $t \in (\theta, u_H)$ implies that there exists $x \in (\theta, u_H)$ such that $\Phi_i(x) > 0$.

Because $\Phi_i(0) = 0$, and for some neighborhood of 0, $G_i(t) > F_i(t)$, Φ_i is decreasing in some neighborhood of 0. Thus, by the same argument as used above, there exists $x \in [0, u_H]$ such that $\Phi_i(x) < 0$. Because it is not the case that $\Phi_i(x) \geq 0$ for all $x \in [0, u_H]$ and also not the case that $\Phi_i(x) \leq 0$ for all $x \in [0, u_H]$, the standard necessary and sufficient condition for second-order stochastic dominance (e.g., Theorem 4.A.7, Shaked and Shanthikumar, 2007) cannot be satisfied by either F_i or G_i.

Part (c). Proposition 2 shows that $G_i(0) > F_i(0)$, $i = S, W$. G_i is flat for $x \in [0, \theta)$. If $F_i(x) \leq G_i(x)$ for all $x \in [0, \theta)$, then because, $G_i(x) = 1 > F_i(x)$, $x \in [\theta, u_H)$ and $G_i(x) = F_i(x) = 1$, for $x \geq u_H$, it would be the case that $F_i(x) \leq G_i(x)$ and, over some interval, $F_i(x) < G_i(x)$. But this is not possible because the mean of F_i and G_i are both equal to the capacity of i. Thus, F_i and G_i must cross at some $x \in (0, \theta)$. Because G_i is flat over this region, only one crossing in $(0, \theta)$ is possible. Thus, for all x sufficiently close to θ, $F_i(x) > G_i(x)$. At the same time $F_i(x) < 1 = G_i(\theta) = 1$ for all $x \in (\theta, u_L)$ and for $x \geq u_L$, $F_i(x) = G_i(x) = 1$. So F_i must cross G_i at θ and does not cross G_i at $x > \theta$. Hence, F_i crosses G_i twice, once at some $x \in (0, \theta)$ and once at $x = \theta$.

For two distributions, $X \overset{d}{\sim} F$ and $Y \overset{d}{\sim} G$, with the same mean such that F crosses G twice, with the first crossing from below, (Theorem 3.1, Klar, 2002) shows that a necessary and sufficient condition for third-order stochastic dominance (called 3-icv in Klar, 2002) is that $\mathbb{E}[X^2] \leq \mathbb{E}[Y^2]$.

Because, X_W and Y_W have the same mean, $\mathrm{Var}(X_W) \leq \mathrm{Var}(Y_W) \Leftrightarrow \mathbb{E}[X_W^2] \leq \mathbb{E}[Y_W^2]$. By part (a), $\mathrm{Var}(X_W) \leq \mathrm{Var}(Y_W) \Leftrightarrow \mathrm{Var}(X_S) \leq \mathrm{Var}(Y_S)$. Thus, the satisfaction of the hypothesis of part (c) implies that $\mathbb{E}[X_W^2] \leq \mathbb{E}[Y_W^2]$ and $\mathbb{E}[X_S^2] \leq \mathbb{E}[Y_S^2]$.

Part (d). First note that, by the argument given above, if $\mathrm{Var}(X_W) > \mathrm{Var}(Y_W)$ then $\mathbb{E}[X_W^2] > \mathbb{E}[Y_W^2]$ and $\mathbb{E}[X_S^2] > \mathbb{E}[Y_S^2]$, which implies, again by Theorem 3.1 in Klar (2002), that X_W does not TSD Y_W and X_S does not TSD Y_S. At the same time, the fact that in some neighborhood of 0, $G_i > F_i$ implies that Y_i cannot TSD X_i, $i = S, W$.

Proof of Corollary 1 Proposition 4 shows that the hypotheses of the corollary imply that the equilibrium configuration is Eq2. Proposition 5 shows that verifying the proposition only requires verifying that the variance of performance under mixed rewards is less than variance of performance under bonus rewards. Because mean performance the same under mixed and bonus rewards, Parts (a) and (c) of Proposition 5 show that Corollary 1 can be verified by showing that $\mathbb{E}[Y_S^2] > \mathbb{E}[X_S^2]$, where, as in Proposition 5, Y_S (X_S) represent performance by S under bonus (mixed) rewards. Let Z_S represent the distribution of S's performance under rank rewards (Lemma 4). Next, note that under the performance distributions specified in Lemma 4,

$$\mathbb{E}[Y_S^2] = \frac{\mu_S}{\theta}, \quad \mathbb{E}[Z_S^2] = \frac{4\,\mu_S^2}{3}.$$

Comparing these two expressions shows that $\mathbb{E}[Y_S^2] > \mathbb{E}[Z_S^2]$ if and only if $(4/3)\,\mu_S < \theta$. Finally note that as $r \to \infty$, X_S converges in distribution to Z_S. Thus for r sufficiently large, $\mathbb{E}[Y_S^2] > \mathbb{E}[X_S^2]$.

B Extension: Option-based bonus compensation

This section completes the analysis of mixed call option and rank rewards sketched in Section 5.2.2. In this section we assumed that the rank-focus parameter $r = 1$. In order to characterize benchmark performance, in Section B.1, we compute equilibrium performance in the absence of rank rewards. In Section B.2, we fill in the steps required to verify the Eq1 equilibrium under option compensation developed in Section 5.2.2 In Section B.3, we derive the conditions for an Eq2 equilibrium in the option compensation context and present a numerical example.

B.1 Option competition benchmark

For the sake of comparisons, we need to specify optimal risk taking in the option setting in the absence of rank rewards. This is fairly straightforward: in the sq-contest, in the absence of rank rewards, a manager's problem is

$$\max_{Z_i \geq 0} \mathbb{E}[\phi(Z_i), \text{ s.t. } \mathbb{E}[Z_i] \leq c_i, \text{ where}$$
$$\phi(z) = \max[\sqrt{z} - \theta, 0], \quad z \geq 0, \; i = S, W.$$

Result B.1. For $z \geq 0$, $\max\left[\sqrt{z} - \theta, 0\right] \leq \frac{1}{4\,\theta}\, z$.

Proof: To show that

$$\frac{1}{4\,\theta}\, z - \left(\sqrt{z} - \theta\right) = \theta + \frac{z}{4\,\theta} - \sqrt{z} \geq 0$$

is equivalent to showing that

$$\left(\theta + \frac{z}{4\,\theta}\right)^2 - z = \frac{(z - 4\,\theta^2)^2}{16\,\theta^2} \geq 0,$$

which is certainly true. □

Result B.1 implies that $\mathbb{E}\big[\max[\sqrt{Z_i} - \theta, 0]\big] \leq \frac{1}{4\,\theta}\, \mathbb{E}[Z_i] = \frac{c_i}{4\,\theta}$. The linear function $\ell(z) = \frac{1}{4\,\theta}\, z$ is an upper support line for ϕ and meets ϕ only at $z_o = 4\,\theta^2$ and 0. Thus, if Z_i a feasible solution to manager's problem, $\mathbb{E}[\phi(Z)] \leq \mathbb{E}[\ell(Z_i)] \leq \frac{c_i}{4\,\theta^2}$.

If Z_i assigns probability $\frac{c_i}{4\,\theta^2}$ to $Z_i = z_o = 4\,\theta^2$ and assigns probability $1 - \frac{c_i}{4\,\theta^2}$ to $Z_i = 0$, $\mathbb{E}[\phi(Z_i)] = \frac{c_i}{4\,\theta^2}$, the upper bound on manager i's payoff and $\mathbb{E}[Z_i] = c_i$. Thus Z_i is feasible and attains the upper bound on i's payoff.

Moreover, any performance distribution that assigns positive weight to points other than z_o and 0 produces a strictly lower payoff. Thus, this performance distribution is the unique, optimal performance distribution in the sq-contest when rank rewards are absent. Hence, in the *option contest* (i.e., the option setting absent rank rewards), the optimal performance distribution for each manager is to submit performance $x_o = \sqrt{z_o} = 2\,\theta$ with probability $\frac{c_i}{4\,\theta^2}$ $(i = S, W)$ and, otherwise, submit zero performance.

B.2 Eq1 equilibria

Recall the exogenous parameter specifications used for the Eq1 equilibrium developed in Section 5.2.2:

$$c_W = 1, \quad c_S = 1\tfrac{37}{40}, \quad \theta = 1\tfrac{19}{24}. \tag{B-1}$$

We will verify, given this parameter specification, the candidate equilibrium performance distributions for S and W stated in Section 5.2.2 are, in fact, equilibrium performance distributions. The candidate distributions of equilibrium performance are presented below for reference:

$$F_W(x) = p_W^0 + (1 - p_W^0)\ \min\left[\left(\frac{x}{u}\right)^2, 1\right], \quad x \geq 0,$$
$$F_S(x) = (1 - p_S^{x_o})\ \min\left[\left(\frac{x}{u}\right)^2, 1\right] + p_S^{x_o}\mathbb{1}_{x_o}(x) \quad x \geq 0,$$
$$x_o = 2\tfrac{1}{4},\ u = \sqrt{v} = \sqrt{3},\ p_W^0 = \tfrac{1}{3},\ p_S^{x_o} = \tfrac{34}{285},$$

where x_o represents the option target and $p_S^{x_o}$ represents the probability weight placed by S on the option target. The comparisons of equilibrium performance in the option setting, under option, rank, and mixed rewards are provided in Section 5.2.2.

As discussed in the main body of the paper, after the transformation of $Z_i = X_i^2$, manager i ($i = S$ or W) faces the following objective function in sq-performance contest (with an expectational constraint) corresponding to the mixed option and rank contest with a non-linear quadratic capacity constraint.

$$F_j^{\text{sq}}(z) + \max[\sqrt{z} - \theta, 0], \qquad j \neq i;$$

and capacity constraint:

$$\mathbb{E}[Z_i] \leq c_i.$$

Candidate equilibrium in the sq-contest We will first verify an equilibrium for the sq-contest in which manager W submits performance equal to 0 with probability $p_W^0 = 1/3$, and, with probability $1 - p_W^0 = 2/3$, randomizes uniformly over $[0, v]$, $v = 3$. Manager S submits performance equal to $z_o = 81/16$ with probability $p_S^{z_o} = 34/285$, and with probability $1 - p_S^{z_o} = 251/285$, randomizes uniformly over $[0, v]$, $v = 3$. After verifying this candidate equilibrium for the sq-contest, we will then use the correspondence developed in Section 5.1 to show that this equilibrium corresponds to the Eq1 equilibrium of the non-linear quadratic capacity-constraint contest developed in Section 5.2.2.

First, we identify the *sq-option target* z_o. In the sq-contest, because, by the definition of an Eq1 equilibrium, manager S targets the bonus, Remark 3 implies that the reward function for S satisfies

$$\Pi_S^{\text{sq}}(z_o) = F_W^{\text{sq}}(z_o) + \max[\sqrt{z_o} - \theta, 0] = \alpha_S + \beta_S\, z_o, \quad z_o > \theta^2, \tag{B-2}$$
$$\Pi_S^{\text{sq}}(z) = F_W^{\text{sq}}(z) + \max[\sqrt{z} - \theta, 0] \leq \alpha_S + \beta_S\, z, \quad \forall z \geq 0. \tag{B-3}$$

Since we aim to verify an Eq1 equilibrium, we shall also have

$$\alpha_S = p_W^0, \qquad \beta_S = \frac{1 - p_W^0}{v}, \tag{B-4}$$

where v is the upper bound for manager W's sq-performance. The capacity constraints for S and W will be given by

$$\begin{aligned} p_S^{z_o}\, z_o + (1 - p_S^{z_o})\, \frac{v}{2} &= c_S, \\ (1 - p_W^0)\, \frac{v}{2} &= c_W. \end{aligned} \tag{B-5}$$

Because W does not chase for bonus, it must be the cases that $v < \theta^2$. By using the first-order condition to find, z_o, the maximum of $1 + (\sqrt{z} - \theta) - (\alpha_S + \beta_S\, z)$ over z, we see that the payoff of manager S at z_o is

$$1 + (\sqrt{z_o} - \theta).$$

If an Eq1 equilibrium exists, it also must be the case that the option target is in the option exercise region, i.e., $\sqrt{z_o} \geq \theta$. In this case, we can express the maximizing value of z, z_o, using support line parameters as follows:

$$z_o = \frac{1}{4\,\beta_S^2}, \quad z_o \geq \theta^2. \tag{B-6}$$

Using the support line characterization of optimal performance distributions (equation (B-2)), and the capacity constraint (equation (B-5)), we see that equilibrium performance distributions must satisfy

$$\begin{aligned} 1 + (\sqrt{z_o} - \theta) &= \alpha_S + \beta_S\, z_o, \\ \left(1 - p_W^0\right) \frac{v}{2} &= c_W. \end{aligned}$$

These expressions, and equations (B-6) and (B-4) imply that

$$v = 4\,(1 - p_W^0)\,(\theta - 1 + p_W^0), \tag{B-7}$$

$$p_W^0 = 1 - \frac{2\,c_W}{v}. \tag{B-8}$$

Because $p_W^0 \in (0, 1)$ and the upper bound for performance in the subthreshold region is θ^2 in the sq-contest, we see that a solution for v of these two equations is a root of the cubic polynomial

$$E(v) = -16\, c_W^2 + 8\, c_W\, \theta\, v - v^3,$$

v satisfying $v > 2\, c_W$ and $v < \theta^2$. Inserting the parameters of the Eq1 candidate equilibrium in Section 5.2.2, $c_W = 1$ and $\theta = 1\frac{19}{24}$, shows that, if $v = 3$, $E(v) = 0$. Moreover, if $v = 3$ then $v > 2\, c_W$ and $v < \theta^2$. Thus, the solution for v is $v = 3$.

Now, we can verify all parameters of the managers' performance distributions using the fact that $v = 3$. From the expression for β_S in equation (B-4), we first have

$$z_o = \frac{1}{4\,\beta_S^2} = \frac{v^4}{16\, c_W^2} = \frac{81}{16}. \tag{B-9}$$

Using the value of z_o fixed by equation (B-9), we see that $z_o > \theta^2$, which verifies equation (B-6).

Equations (B-4), (B-5), and (B-9) imply that

$$p_W^0 = \frac{1}{3}, \quad p_S^{z_0} = \frac{34}{285}.$$

Finally, the multipliers follow,

$$\alpha_S = p_W^0 = \frac{1}{3}, \qquad \beta_S = \frac{1 - p_W^0}{v} = \frac{2}{9}, \qquad \beta_W = \frac{1 - p_S^{z_o}}{v} = \frac{251}{855}.$$

To complete the verification of candidate sq-contest equilibrium, we need only show that W will not defect from the equilibrium to chase option compensation. First, note that $\beta_W \geq \beta_S$. Otherwise we have $\alpha_S + \beta_S\, z > \beta_W\, z$ for all z, in which case, $\beta_W\, z < \alpha_S + \beta_S\, z = \Pi_S^{\text{sq}}(z_o+) = \Pi_W^{\text{sq}}(z_o+)$, contradicting the multiplier condition.

Manager W has two viable defection strategies: (i) targeting a different option target than S or (ii) just topping S's target, z_o. Under strategy (i), W will never gain from a higher option target because, from the first-order condition for maximizing $\beta_W\, z - (\sqrt{z} - \theta)$, the maximum obtained, $\frac{1}{4\,\beta_W^2} \leq \frac{1}{4\,\beta_S^2} = z_o$. Hence, conditional on targeting the bonus but not topping S's maximum performance, z_o, W's best option is $\frac{1}{4\,\beta_W^2}$ if $\frac{1}{4\,\beta_W^2} > \theta^2$, and θ^2 otherwise, i.e., $z_W = \max[\frac{1}{4\,\beta_W^2}, \theta^2]$. Therefore, the condition for W not defecting to a performance level within $[\theta^2, z_o)$ is

$$\beta_W\, z_W \geq (1 - p_S^{z_o}) + (\sqrt{z_W} - \theta). \tag{B-10}$$

Under strategy (ii), i.e., if W instead attempts to just top S, the payoff to W will equal $\Pi_W^{\text{sq}}(z_o+)$, Note that $\Pi_W^{\text{sq}}(z_o+) = \Pi_S^{\text{sq}}(z_o+)$. The fact that S is targeting z_o implies that $\Pi_S^{\text{sq}}(z_o+) = \ell_S(z_o)$ Thus, the condition for W not defecting to just topping S's option target, z_o, is

$$\alpha_S + \beta_S\, z_o \leq \beta_W\, z_o. \tag{B-11}$$

Using the parameters in the candidate equilibrium for the sq-contest, we see equations (B-10) and (B-11) are satisfied.

Thus, we have verified the candidate equilibrium for the sq-contest. This equilibrium in sq-performance corresponds to an equilibrium in the non-linear quadratic-constraint contest though the correspondence detailed in Section 5.1. In the non-linear quadratic-constraint contest, the upper bound on performance in the subthreshold region, u, the option target for S, x_o, the probability that W targets 0, p_W^0, the probability that S targets the option, $p_S^{x^o}$, and the option target, x_o, are given by the following correspondence: p_W^0 is the same in both the sq-contest and the non-linear quadratic-constraint contest, $p_S^{x_o} = p_S^{z_o}$, and

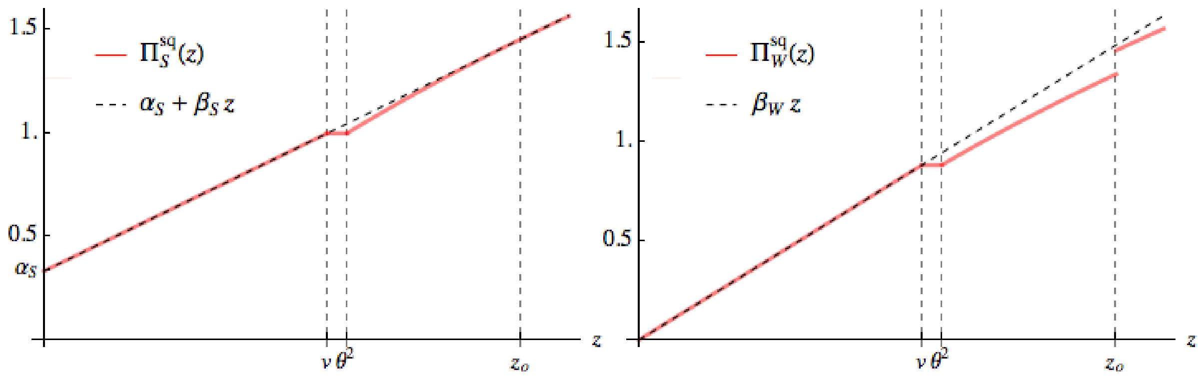
$$\begin{cases} u = \sqrt{v}, \\ x_o = \sqrt{z_o}, \\ Z \overset{d}{\sim} \text{Unif}[0, v] \Longleftrightarrow X = \sqrt{Z} \overset{d}{\sim} \min\left[\left(\dfrac{x}{u}\right)^2, 1\right], x \geq 0. \end{cases}$$

Thus, the Eq1 equilibrium in Section 5.2.2 has been verified.

Figure B-1 illustrates the reward functions and support lines for the equilibrium of the sq-contest.

B.3 Eq2 equilibria

In this section, we construct an Eq2 equilibrium for mixed option and rank contests. Again recall the simplifying assumption made in Section 5.2.2 that $r = 1$. In the option compensation setting, an Eq2 equilibrium is an equilibrium in which S and W compete for rank



A. Manager S B. Manager W

Figure B-1: *Equilibrium rewards and support lines in the sq-contest under option compensation.* The figure illustrates the sq-reward functions, Π_i^{sq}, and support lines, $\alpha_i + \beta_i z$, for manager $i = S$ (Panel A) and manager $i = W$ (Panel B). Over the region $[0, v]$ in sq-performance space, the managers compete for rank rewards below the option exercise performance threshold; manager S targets performance z_o above the option exercise threshold. Manager W does not attempt performance in excess of the option exercise threshold. The horizontal axis represents sq-performance, z. The parameters used in the figure are $\alpha_S = 1/3$, $\beta_S = 2/9$, $\beta_W = 251/855$, $v = 3$, $\theta = 119/24$, $z_o = 81/16$, $p_W^0 = 1/3$, and $p_S^{x_o} = 34/285$.

dominance both over an interval $[0, u]$, where u is less than the option exercise threshold, and over some interval $[x_o, x_H]$, where x_o is greater than the option exercise threshold.

As in Section B.2, we will present conditions for the existence of an Eq2 equilibrium in sq-contest that corresponds to an Eq2 equilibrium in the non-linear quadratic-constraint contest specified in Section 5.2.2. In the sq-contest, the interval of subthreshold rank competition is $[0, v]$, where $v = u^2$ and the interval of superthreshold competition is $[z_o, z_H]$, were $z_o = x_o^2$ and $z_H = x_H^2$.

In the sq-contest, we need to ensure that the upper bond of the subthreshold region is less than the minimum sq-performance required to capture option rewards, and that sq-performance at the lower bound of the superthreshold region exceeds the minimum performance required to capture option rewards, i.e.,

$$v < \theta^2 \quad \text{and} \quad z_o > \theta^2. \tag{B-12}$$

Let G_S and G_W represent the *sq-performance distributions* for S and W. We aim to verify an Eq2 configuration equilibrium, thus

$$G_S(z) = (1 - p_S^h - p_S^{z_o}) \min\left[\frac{z}{v}, 1\right] + p_S^{z_o} \mathbb{1}_{z_o} + p_S^h G_S^h(z), \tag{B-13}$$

$$G_W(z) = p_W^0 + (1 - p_W^0 - p_W^h) \min\left[\frac{z}{v}, 1\right] + p_W^h G_W^h(z). \tag{B-14}$$

where G_i^h is a continuous distribution chosen by manager $i = S, W$ conditioned on choosing performance in (z_o, z_H). As we shall see, in the option compensation setting, generally, $G_S^h \neq G_W^h$, however, since because distributions have the same support, $G_S(x_o) = G_W(x_o) = 0$ and $G_S(x_H) = G_S(x_H) = 1$.

In order for Π_S^{sq} to meet the support line over the option competition region, $[z_o, z_H]$, it must be the case that

$$\left[(1-p^h_W)+p^h_W\,G^h_S(z)+(\sqrt{z}-\theta)\right]-(\alpha_S+\beta_S\,z)=0.$$

Thus,

$$G^h_S(z)=\frac{\alpha_S+z\,\beta_S-\left(1-p^h_W+\sqrt{z}-\theta\right)}{p^h_W},\quad \text{if } z\in[z_o,z_H], \tag{B-15}$$

and $G^h_S(z)=0$, if $z<z_o$ and $G^h_S(z)=1$, if $z>z_H$. Because G^h_S must be increasing, by using the first-order condition we find that G^h_S is increasing for $z\geq z_o$ if and only if

$$z_o\geq 1/(4\,\beta^2_S). \tag{B-16}$$

Similarly, we have

$$G^h_W(z)=\frac{z\,\beta_W-\left(1-p^h_S+\sqrt{z}-\theta\right)}{p^h_S},\quad \text{if } z\in[z_o,z_H], \tag{B-17}$$

and $G^h_W(z)=0$, if $z<z_o$ and $G^h_W(z)=1$, if $z>z_H$. Because, as an argument very similar to the argument used in the Eq1 case developed above shows, $\beta_W<\beta_S$ in an Eq2 configuration, Equation (B-16) also ensures that G^h_W is increasing.

We split the derivation of the necessary and sufficient conditions for an Eq2 equilibrium into a series steps for the sake of readability.

(i) *Subthreshold rank competition.* Remark 3 implies that the slope of the reward functions for both managers over the supports of their performance distributions are the same over the subthreshold and superthreshold competition regions. Over the subthreshold region,in Eq2 equilibria, the relations between the support lines and the distributional parameters are the same in the baseline (bonus competition) and the option compensation settings, i.e.,

$$\alpha_S=p^0_W,\qquad \beta_S=\frac{1-p^0_W-p^h_W}{v},\qquad \beta_W=\frac{1-p^{z_o}_S-p^h_S}{v}. \tag{B-18}$$

(ii) *Conditional distributions at z_o.* At z_o, it must be the case that $G^h_S(z_o)=G^h_W(z_o)=0$. Using equation (B-15) and equation (B-17), we see that $G^h_S(z_o)=G^h_W(z_o)=0$ implies that

$$\alpha_S+z_o\,\beta_S-\left(1-p^h_W+\sqrt{z_o}-\theta\right)=z_o\,\beta_W-\left(1-p^h_S+\sqrt{z_o}-\theta\right)=0. \tag{B-19}$$

Equation (B-19) and the expressions of multipliers in an Eq2 configuration (equation (B-18)) imply that

$$z_o=\frac{\alpha_S-(p^h_S-p^h_W)}{\beta_W-\beta_S}=v\,\frac{p^0_W-(p^h_S-p^h_W)}{(p^0_W+p^h_W)-(p^h_S+p^{z_o}_S)}. \tag{B-20}$$

(iii) *Equality at the upper bound of the conditional distributions.* At z_H, it must be the case that $G^h_S(z_H)=G^h_W(z_H)$. Note that, at z_H, the reward to both S and W is the same and that, since z_H is in the support of both S's and W's performance distributions, the reward at z_H equals the support line evaluated at z_H (Remark 3), i.e., $\ell_S(z_H)=\Pi^{\text{sq}}_S(z_H)=\Pi^{\text{sq}}_W(z_H)=\ell_S(z_H)$. Using this fact, and the multipliers (Equation (B-18)) we see that

$$z_H=\frac{\alpha_S}{\beta_W-\beta_S}=v\,\frac{p^0_W}{(p^0_W+p^h_W)-(p^h_S+p^{z_o}_S)}. \tag{B-21}$$

Because $v < z_o < z_H$ in an Eq2 equilibrium, we see that equations (B-21) and (B-20) impose the following restrictions on the distributional parameters for an Eq2 equilibrium:

$$(p_W^0 + p_W^h) - (p_S^h + p_S^{z_o}) > 0, \tag{B-22}$$

$$p_S^h - p_W^h > 0, \tag{B-23}$$

(iv) *Restrictions imposed by the reward function's intersection with the support line at* $z = z_o$. Consider the region $z \in [v, z_o]$. The following expression, representing the difference between the reward function and the support line for S,

$$\Pi_S^{\mathrm{sq}}(z) - \alpha_S - \beta_S\, z = (1 - p_W^h) + \max[\sqrt{z} - \theta, 0] - \alpha_S - \beta_S\, z$$

is clearly decreasing when $z < \theta^2$, and is concave when $z \in [\theta^2, z_o]$. Note that Π_S^{sq} must approach the support line from below (Remark 3). Because G_W does not jump up at θ, we must have $z \hookrightarrow (1 - p_W^h) + \max[\sqrt{z} - \theta, 0] - \alpha_S - \beta_S\, z$ increasing in order for the reward function to meet the support line. Thus, it must be the case that $z_o \leq 1/(4\,\beta_S^2)$. Combining this result with equation (B-16) and using the multipliers (equation (B-18)) shows that

$$z_o = \frac{1}{4\,\beta_S^2} = \frac{v^2}{4\,(1 - p_W^0 - p_W^h)^2}. \tag{B-24}$$

Noting equation (B-24) and the expression of z_o given by equation (B-20), one can solve for v to obtain

$$v = 4\,(1 - p_W^0 - p_W^h)^2\, \frac{p_W^0 - (p_S^h - p_W^h)}{(p_W^0 + p_W^h) - (p_S^h + p_S^{z_o})}. \tag{B-25}$$

Noting the definition of G_S^h (equation (B-15)), and using equation (B-18) and equation (B-24), shows that $G_S^h(z_o) = 0$ implies that

$$\theta = 1 - p_W^0 - p_W^h + \frac{v}{4\,(1 - p_W^0 - p_W^h)}. \tag{B-26}$$

(v) *Fixing the upper endpoint of the conditional distributions.* At z_H, $G_S^h(z_H) = 1$. Because $G_S^h(z_H) = 1$, the definition of G_S^h (equation B-15) implies that

$$\frac{\alpha_S + z\,\beta_S - \left(1 - p_W^h + \sqrt{z} - \theta\right)}{p_W^h} = 1. \tag{B-27}$$

Using equations (B-18) and (B-21) we see that equation (B-27) implies that

$$1 - p_W^0 - 2\,p_W^h + \frac{(p_S^{z_o} + p_W^0)\,(1 - p_W^0 - p_W^h)}{(p_W^0 + p_W^h) - (p_S^h + p_S^{z_o})} = 2\sqrt{\frac{p_W^0\,(p_W^0 + p_W^h - p_S^h)\,(1 - p_W^0 - p_W^h)^2}{\left((p_W^0 + p_W^h) - (p_S^h + p_S^{z_o})\right)^2}}. \tag{B-28}$$

Step (ii) and this step ensure that it is also the case that $G_W^h(z_H) = 1$.

(vi) *Imposing the capacity constraints* Inspecting the definitions of G_i and G_i^h, $i = S, W$ (equations (B-13), (B-14), (B-15), and (B-17)) shows that the capacity constraints will be satisfied if and only if

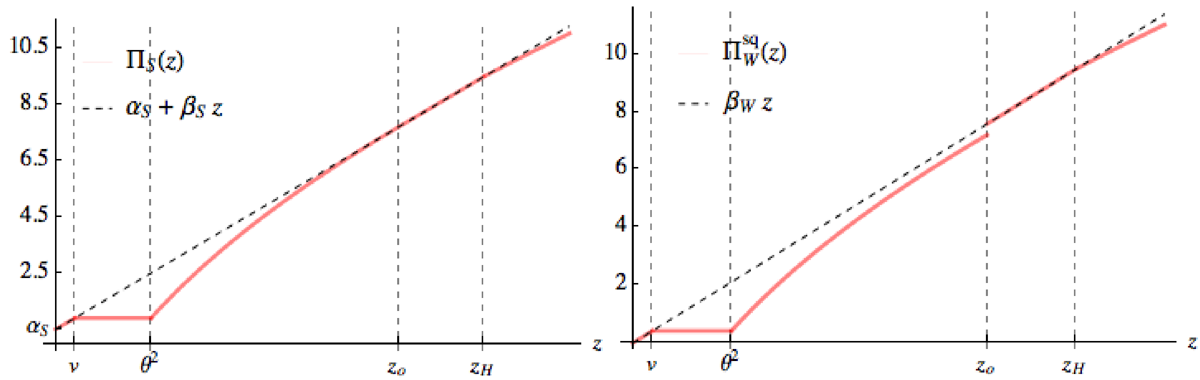
$$(1 - p_S^h - p_S^{z_o})\,\frac{v}{2} + p_S^{z_o}\, z_o + p_S^h \int_{z_o}^{z_H} z\, g_S^h(z)\, dz = c_S, \tag{B-29}$$

$$(1 - p_W^h - p_W^0)\,\frac{v}{2} + p_W^h \int_{z_o}^{z_H} z\, g_W^h(z)\, dz = c_W, \tag{B-30}$$

where g_i^h is the density of G_i^h, $i = S, W$.

Thus, the parameters $p_W^0 \in (0,1)$, $p_S^{z_o} \in (0,1)$, $p_S^h \in (0,1)$, $p_W^h \in (0,1)$, $v > 0$, $z_o > 0$, and $z_H > z_o$ support an Eq2 equilibrium if and only if the seven equations (B-20), (B-21), (B-25), (B-26), (B-28), (B-29), and (B-30) are satisfied. In addition, the parameters must satisfy inequalities (B-12), (B-22), (B-23), and satisfy $p_S^h + p_S^{z_o} < 1$ and $p_W^h + p_W^0 < 1$.

As one might expect given the complexity of the conditions, we are not able to provide an algebraic solution for the Eq2 configuration. However, the equations are numerically solvable. Using this numerical solution we present, in Figure B-2, an example of the reward functions and support line (in sq-performance space) of an Eq2 equilibria. The distributional parameters are presented in the caption to the figure.



A. Manager S B. Manager W

Figure B-2: *Equilibrium rewards and support lines in Eq2 equilibria under option compensation.* The figure illustrates the sq-reward functions, $\Pi_i^{\rm sq}$ and support lines, $\alpha_i + \beta_i z$ for manager S (Panel A) and manager W (Panel B). The horizontal axis represents sq-performance, z. The exogenous parameters are $c_S = 127.40$, $c_W = 26.29$, and $\theta = 7.58$. The parameters of the solution are $p_W^0 = 0.5$, $p_W^h = 0.1$, $p_S^h = 0.2$, $p_S^{z_o} = 0.38$, $z_o = 206.1$, $z_H = 257.6$, and $v = 11.48$. Over the region $[0, v]$ in sq-performance space the manager compete for rank rewards below the option exercise threshold; over the region $[z_o, z_H]$ the managers compete for rank rewards above the option exercise threshold.

Because the probability of ruin risk taking by W is the same in the sq-contest and the non-linear quadratic constraint contest, we see that, in this Eq2 equilibrium with mixed rewards, the probability of ruin risk taking is $p_W^0(\text{mixed}) = 0.50$. Using the characterization of ruin risk taking developed in Section B.1, we can compute ruin risk taking in the option contest benchmark setting, where rank rewards are absent. This computation yields, $p_W^0(\text{opt}) = 0.886$, Using our characterization of rank competitions in the absence of absolute performance rewards (Lemma 4), we see that, in a rank competition, the probability of ruin risk taking for W is $p_W^0(\text{rnk}) = 0.793$. Thus, as in the baseline model, which assumes bonus compensation, in this Eq2 equilibrium under option compensation, ruin risk taking under mixed rewards is considerably less than ruin risk taking under either absolute performance rewards or rank rewards.

C Extension: Assured bonus compensation

In this section, we characterize equilibria when bonus packages are assured, i.e., $\mu_S > \theta$. Our first result is that Eq2 equilibrium configurations can sometimes be sustained when $\mu_S > \theta$. See the necessary and sufficient condition provided by Lemma C-1.

Lemma C-1 *When $\mu_S > \theta$, No Eq1 or Eq0 equilibria exist; Eq2 equilibria exist if and only if*

$$\mu_S \leq \frac{\theta}{r}\left((1+r)\sqrt{1+\frac{\mu_W^2\, r^2}{\theta\,(\theta+2\,\mu_W\, r)}}-1\right). \tag{C-1}$$

Proof: It is obvious that no Eq0 equilibria exist because S's capacity exceeds the bonus threshold, θ and, in an Eq0 equilibria, the capacity constraint is binding. Now, consider Eq1 equilibria when $\mu_S > \theta$. The definitions of equilibrium performance distributions (Lemma 3) imply that

$$1-p_S^\theta = 1-\frac{2\,\mu_S-u}{2\,\theta-u} = \frac{(\theta-\mu_S)\left(\mu_W+\sqrt{\mu_W\,(2\,\theta/r+\mu_W)}\right)}{\theta\,\sqrt{\mu_W\,(2\,\theta/r+\mu_W)}} < 0.$$

Thus, no Eq1 equilibrium exists.

Now consider Eq2 equilibria. First, we prove the necessity of condition (C-1). For Eq2 equilibrium to be verified, we must have $u_L \geq 0$. By the definition in equation (A-21) from Lemma 1, u_L has the same sign as

$$\theta\ (\theta-\mu_S)\ (\theta+2\,\theta/r+\mu_S)+2\,r\ (\theta-\mu_S)\ (\theta+2\,\theta/r+\mu_S)\ \mu_W+(1+r)^2\,\theta\,\mu_W^2.$$

Define $x \coloneqq \mu_S-\theta$. The above expression becomes quadratic in x:

$$-r\,(\theta/r+2\,\mu_W)\,x^2-2\,(1+r)\,\theta\,(\theta/r+2\,\mu_W)\,x+(1+r)^2\,\theta\,\mu_W^2.$$

The positive root of this expression is given by

$$x=\mu_S-\theta \leq \frac{\theta\,(1+r)}{r}\left(\sqrt{1+\frac{\mu_W^2}{\theta/r\,(\theta/r+2\,\mu_W)}}-1\right),$$

i.e., inequality (C-1) in the lemma.

Next, we prove the sufficiency, it is sufficient to show that, if condition (C-1) is satisfied, the constraints on other parameters can be verified.

Start with p_S^θ. From equation (A-18), for $p_S^\theta < 1$ we need to show that $\mu_S < \theta+\mu_W+\mu_W\, r$. Condition (C-1) is sufficient for this inequality to hold because

$$\theta+\mu_W+\mu_W\, r-\frac{\theta}{r}\left((1+r)\sqrt{1+\frac{\mu_W^2\, r^2}{\theta\,(\theta+2\,\mu_W\, r)}}-1\right) = \frac{(1+r)\,(\theta+\mu_W\, r)\left(1-\frac{\theta}{\sqrt{\theta\,(\theta+2\,\mu_W\, r)}}\right)}{r} > 0$$

guarantees that $\theta+\mu_W+\mu_W\, r-\mu_S > 0$.

Next, we show that $u_H > \theta$. From definition (A-20), $u_H - \theta$ has the same sign as

$$(\theta - \mu_S)^2(\theta/r + \mu_W)^2 - \mu_W\,(2\,\theta/r + \mu_W)\,(\theta/r + \mu_S)^2, \tag{C-2}$$

which is a convex function of μ_S defined over $\mu_S \in [\theta, \overline{\mu}_S]$, where

$$\overline{\mu}_S := \frac{\theta}{r}\left((1+r)\sqrt{1 + \frac{\mu_W^2\, r^2}{\theta\,(\theta + 2\,\mu_W\, r)}} - 1\right). \tag{C-3}$$

Expression (C-2) is negative at $\mu_S = \theta$. Letting $\mu_S = \overline{\mu}_S$, we see that the expression has the same sign a $1 + \frac{\theta}{\theta + 2\,\mu_W\, r} - 2\sqrt{\frac{(\theta+\mu_W\, r)^2}{\theta\,(\theta+2\,\mu_W\, r)}}$, which has the same sign as

$$\left(1 + \frac{\theta}{\theta + 2\,\mu_W\, r}\right)^2 - \left(2\sqrt{\frac{(\theta + \mu_W\, r)^2}{\theta\,(\theta + 2\,\mu_W\, r)}}\right)^2 = -\frac{8\,\mu_W\,(\theta + \mu_W\, r)^2}{\theta\,(\theta + 2\,\mu_W\, r)^2} < 0.$$

Thus expression (C-2) is negative at $\mu_S = \overline{\mu}_S$ as well. Because the maxima of a convex function are attained at its extreme points, $u_H > \theta$. The satisfaction of the constraints on p_S^θ, p_W^0, p_S^h and p_W^h follow in like fashion, using the definitions provided by (A-18) and (A-19). $\square$

When the upper bound on μ_S defined by equation (C-3) is not satisfied, new equilibrium configurations can be verified. Three new configurations can be realized for some choices of model parameters.

1. Eq0B: manager W randomizes between 0 and a uniform distribution over the superthreshold region; manager S randomizes between a point mass at θ and a uniform distribution over the superthreshold region.
2. Eq0θ: manager W randomizes between 0 and θ, and manager S captures both the rank and bonus reward with probability 1. In the Eq0θ configuration, S's strategies are not uniquely determined and the capacity constraint for S need not bind; but, if, for any given parameter choice, any performance strategy verifies an Eq0θ equilibrium, uniform randomization by S over the superthreshold region verifies an Eq0θ equilibrium.
3. EqBB: manager W places point mass at θ and uniformly randomizes over the superthreshold region; manager S uniformly randomizes over superthreshold region.

Lemma C-2 (Eq0B, Eq0$\boldsymbol{\theta}$ and EqBB)

(i) When $\mu_W \le \theta < \mu_S$,

(a) Eq0B equilibria exist if and only if $\frac{\theta}{r}\left((1+r)\sqrt{1 + \frac{\mu_W^2\, r^2}{\theta\,(\theta+2\,\mu_W\, r)}} - 1\right) < \mu_S < \theta + \frac{\theta\, r}{2}$, and can be characterized by the following distributions:

$$F_S^* = p_S^\theta\, \mathbb{1}_\theta + (1 - p_S^\theta)\,\mathrm{Unif}[\theta, u], \quad F_W^* = p_W^0\, \mathbb{1}_0 + (1 - p_W^0)\,\mathrm{Unif}[\theta, u].$$

(b) Eq0θ equilibria exist if and only if $\mu_S \ge \theta + \frac{\theta\, r}{2}$, and can be characterized by the following distributions:

$$F_S^* = \mathrm{Unif}[\theta, 2\,\mu_S - \theta], \quad F_W^* = p_W^0\, \mathbb{1}_0 + (1 - p_W^0)\,\mathbb{1}_\theta.$$

Other Eq0θ equilibria exist in which S chooses a different performance distribution, whose support is also contained in $[\theta, \infty)$. These equilibria are payoff equivalent.

(ii) When $\theta < \mu_W$, EqBB equilibria exist if and only if $\mu_S \geq \theta + \frac{\theta\, r}{2}$, and can be characterized by the following performance distributions:

$$F_S^* = \mathrm{Unif}[\theta, 2\,\mu_S - \theta], \quad F_W^* = p_W^\theta\, \mathbb{1}_\theta + (1 - p_W^\theta)\, \mathrm{Unif}[\theta, 2\,\mu_S - \theta].$$

All the parameters are specified in the proof. In all the equilibria, W's capacity constraint binds; In all equilibria, except perhaps Eq0θ equilibria, S's capacity constraint binds.

PROOF:

(ia) For Eq0B, the capacity constraint for S is $p_S^\theta\, \theta + (1 - p_S^\theta)\,(\theta/2 + u/2) = \mu_S$. The slope constraint for W, produced by S's distribution, is $\frac{1-p_S^\theta}{u-\theta} = \frac{1+r}{r\,u}$. The two constraints jointly determine the value pair

$$p_S^\theta = \frac{\mu_S - \sqrt{(\mu_S - \theta)\,(\theta + 2\,\theta/r + \mu_S)}}{\theta}, \quad u = \frac{\theta + r\,\mu_S + r\,\sqrt{(\mu_S - \theta)\,(\theta + 2\,\theta/r + \mu_S)}}{1 + r} > \theta.$$

To show that $p_S^\theta < 1$ is equivalent to show that $(\mu_S - \theta)^2 < (\mu_S - \theta)\,(\theta + 2\,\theta/r + \mu_S)$, which holds for sure. For $p_S^\theta > 0$, we need $\mu_S^2 - (\mu_S - \theta)\,(\theta + 2\,\theta/r + \mu_S) = \theta\,(\theta + 2\,\theta/r - 2\,\mu_S/r)$ to be positive. Thus we need the following condition:

$$\mu_S \leq \theta + \frac{\theta\, r}{2}.$$

The capacity constraint of W is $(1 - p_W^0)\,(\theta/2 + u/2) = \mu_W$, which implies that

$$p_W^0 = \frac{\theta + u - 2\,\mu_W}{\theta + u}. \tag{C-4}$$

$p_W^0 \in [0, 1]$ because $u > \theta > \mu_W$. We also need $p_W^0 \leq (1 - 1/r) - (1 - p_W^0)\,\frac{u}{u-\theta}$ to insure that S's support line lies above $(0, p_W^0)$. Thus we check

$$(1-1/r)-(1-p_W^0)\,\frac{u}{u-\theta}-p_W^0 = \frac{r\,\left(\theta/r + \mu_S + \sqrt{(\mu_S - \theta)\,(\theta + 2\,\theta/r + \mu_S)}\,\right)^2}{(1+r)^2} - \theta\,(\theta/r + 2\,\mu_W),$$

which is nonnegative if

$$\mu_S \geq \frac{\theta}{r}\left((1+r)\,\sqrt{1 + \frac{\mu_W^2\, r^2}{\theta\,(\theta + 2\,\mu_W\, r)}} - 1\right).$$

Conditional on performance not equal to zero, the expected performance of the weak manager is greater than it is under bonus rewards. Because in both cases, the capacity constraint for W binds, ruin risk taking in any Eq0B equilibrium is higher than ruin risk taking under bonus rewards.

(ib) For Eq0θ, if S randomizes uniformly over a superthreshold region $[\theta, u]$, then her capacity constraint implies that $u = 2\,\mu_S - \theta$. Thus the slope of W's payoff function over $[\theta, u]$ equals $\frac{1}{2\,(\mu_S - \theta)}$. The slope of W's support line if W simply chases the bonus equals, $\frac{1}{\theta\, r}$. For not pursuing rank competition to be a best reply for W we need

$$\frac{1}{\theta\, r} \geq \frac{1}{2\,(\mu_S - \theta)}.$$

This holds if $\mu_S \geq \theta + \theta\, r/2$. In the Eq0$\theta$ managerial strategies are identical to the strategies the managers would play in the absence of rank rewards. Thus, rank rewards have no effect on ruin risk taking.

(ii) For EqBB, similarly, the superthreshold region $[\theta, u]$ has a support $u = 2\,\mu_S - \theta$. manager W's capacity constraint $p_W^\theta\,\theta + (1 - p_W^\theta)\,\mu_S = \mu_W$ uniquely determines the point mass on bonus threshold, p_W^θ. From the assumption $\mu_S > \theta$, we conclude that $\mu_W > \theta$.

The support line for S is

$$\left(1 + \frac{1}{r}\right) - (1 - p_W^\theta)\left(\frac{x - \theta}{u - \theta}\right);$$

and for W is

$$\left(1 + \frac{1}{r}\right) - \frac{1}{u - \theta}\,(u - x).$$

Because the support lines must lie above the origin, condition $\mu_S \geq \theta + \theta\, r/2$ must hold to sustain the EqBB equilibrium.

As there is no point mass at 0 for EqBB configuration, albeit for the completely obvious reason that bonus compensation is so large and so easy to capture that it does not motivate risk taking. In the absence of rank rewards there also would be no mass placed on 0. Thus the introduction of rank rewards has no effect on ruin risk taking. □

D Extension: Different bonus thresholds

Consider a setting with two bonus thresholds, denoted by θ_i for manager i ($i = S, W$). A necessary condition for such a configuration to be verified is that the managers' reward function intersects the support line at the manager's respective bonus thresholds, i.e.,

$$1 + r = \alpha_S + \beta_S\,\theta_S, \quad r\,(1 - p_S^\theta) + 1 = \beta_W\,\theta_W, \tag{D-1}$$

In equation (D-1), the left-hand side of each equation is the payoff from targeting the bonus and the right-hand side is the support line for the respective managers evaluated at the bonus threshold. The other necessary condition for an equilibrium with this configuration is that the support line intersects the reward function for both managers in the region over which the managers compete for rank dominance, i.e.,

$$r\,p_W^0 + r\,(1 - p_W^0 - p_W^\theta)\frac{x}{u} = \alpha_S + \beta_S\,x, \quad r\,(1 - p_S^\theta)\frac{x}{u} = \beta_W\,x, \quad x \in [0, u], \tag{D-2}$$

where the multipliers, α_W and β_W and β_S, are defined as in Section 3. To construct our example, we assume $r = 1$. Using (D-1) and (D-2) we see that

$$\theta_S = \frac{2 - p_W^0}{1 - p_W^0 - p_W^\theta}\,u, \qquad \theta_W = \frac{2 - p_S^\theta}{1 - p_S^\theta}\,u.$$

Comparing the two thresholds, we see that $(1 + p_W^\theta)\,(1 - p_S^\theta) - (1 - p_W^0 - p_W^\theta) > 0$ and thus, in fact, $\theta_S > \theta_W$, i.e., the weak manager's bonus threshold is lower than the strong managers' threshold.

An example of this equilibrium configuration is provided in Figure D-1. The statistics comparing ruin risk taking in multi-target setting, under bonus, rank, and mixed rewards are provided in Table 1.

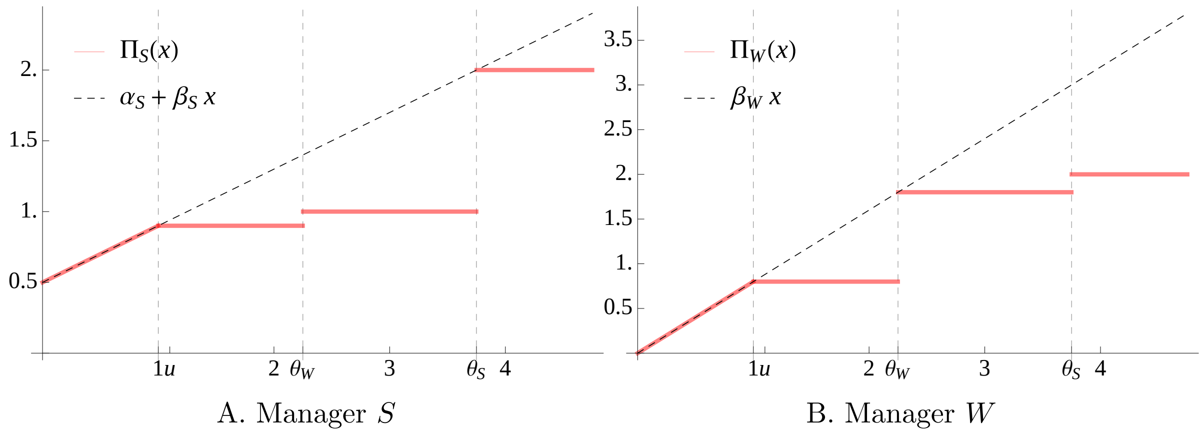


Figure D-1: *Equilibrium rewards and support lines, different thresholds.* The figure illustrates the reward functions, Π_i and support lines, $\alpha_i + \beta_i\,x$ for manager S (Panel A) and manager W (Panel B). The horizontal axis represents performance, x. The parameters are $\theta_S = 3.75$, $\theta_W = 2.25$, $\mu_S = 1.15$, $\mu_W = 0.425$, $p_S^\theta = 0.2$, $p_W^\theta = 0.1$, $p_W^0 = 0.5$.

Ruin Risk	Bonus rewards	Rank rewards	Mixed rewards
p_W^0	0.811	0.630	0.500

Table 1: *Statistics for ruin risk taking under multiple bonus targets.*

E Extension: Multi-manager equilibria

E.1 Marginal return from rank-competition when the number of managers exceeds 2

The two fundamental drivers of the results in our baseline model with two managers were (a) rank rewards cause managers to spread out their performance, and this effect reduce the propensity of managers to accept ruin risk in the pursuit of absolute performance rewards, (b) the pursuit of absolute performance rewards is relatively more appealing to stronger managers and thus the disproportionate diversion of stronger managers' capacity to the pursuit of absolute performance rewards reduces the need for weaker managers to accept ruin risk to compete for rank dominance. In the baseline two manager setting, these two drivers ensure that mixed rewards produce less ruin risk taking than either pure rank or pure absolute performance rewards.

The question we address in this section is how robust are these drivers to introducing more managers into a contest. There is no reason to suspect that the first driver, (a), is affected by the number of managers. As shown in Section 2.1, the basis for this driver is simply that it is very profitable in a rank competitor to "just top" the performance of a rival when that rival's performance is concentrated on a point or small interval of performance levels. The explanation for this effect, developed in Section 2.1, applies without modification to contests with many managers.

It is perhaps less obvious that the second driver, (b), also operates in contests with many managers. This driver is based on stronger managers, in rank competitions, having a lower marginal benefit from increased performance capacity than weaker managers. Stronger managers' lower marginal benefit from using capacity to pursue rank dominance, ensure that stronger managers will divert more capacity to chasing absolute performance rewards. This effect reduces the ruin risk that weaker manager will accept when competing with stronger managers and thus lowers overall ruin risk taking.

In this section we will show that when there are a finite number of managers competing in a rank competition, the marginal gain from increased capacity is always at least weakly decreasing in the strength (i.e., performance capacity) of the managers. Thus, driver (b) operates in multi-manager competitions. Next we will characterize an example of a multi-manager equilibrium in which ruin risk taking is less under mixed rewards than under either pure absolute or pure rank rewards. In this example, absolute performance rewards are bonus rewards.

Consider a risk taking contest, with $n \geq 2$ managers, assume that capacity is decreasing in the index of the manager, i.e., $\mu_1 > \mu_2 > \mu_3 \ldots \mu_{n-1} > \mu_n > 0$. Let $\boldsymbol{\mu} = (\mu_1, \mu_2 \ldots \mu_n)$ represent the vector of capacities.

Assume, as in the manuscript, that managers submit non-negative random variables, $\tilde{x}_i$, $i \in \{1, 2, \ldots n\}$. Represent arbitrary realizations of $\tilde{x}_i$ with x. We will call these realizations "performance." The random variables are constrained by a capacity constraint—i.e., their expectation is no greater than capacity, μ_i, $\mathbb{E}[\tilde{x}_i] \leq \mu_i$. The manager with the highest performance receives a rank reward of 1, and all other managers receive a rank reward of 0.[1] We call such a contest a $\boldsymbol{\mu}$*-contest*. In a given equilibrium of the $\boldsymbol{\mu}$*-contest*, let $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ be the associated vectors of multipliers, and let $\boldsymbol{F}$ represent the vector of manager performance distributions.

Now consider an all-pay auction in which bidders bid x for an auctioned good. The highest bidder receives the good. All bidders, the winner as well as the losers, pay their bids to the auctioneer. Each of the n bidders has a positive valuation of the good, v_i, $i = 1 \ldots n$. Let $\boldsymbol{v} = (v_1, v_2 \ldots v_n)$ be the vector of such values. We call an all-pay auction where bidders' valuations are given by $\boldsymbol{v}$ a $\boldsymbol{v}$*-auction.* Let $\boldsymbol{G}$ represent the vector of manager bid distributions in the auction and let $\boldsymbol{u}^* = (u_1^*, u_2^* \ldots u_n^*)$ represent the vector of manager payoffs given $\boldsymbol{G}$ and $\boldsymbol{v}$.

Lemma E-1 *Suppose that* $\boldsymbol{F}$ *is a* $\boldsymbol{\mu}$*-contest equilibrium with associated multipliers* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$*. If*

$$v_i = \frac{r}{\beta_i} \text{ and } u_i^* = \frac{\alpha_i}{\beta_i} \implies \boldsymbol{F} \text{is a } \boldsymbol{v}\text{-auction equilibrium},$$

and, in this $\boldsymbol{v}$*-auction equilibrium,* $E[\tilde{x}_i] = \mu_i$*.*

Proof: The condition for a best reply in the $\boldsymbol{\mu}$-contest is that for all i,

$$\begin{aligned} x \in \text{Supp}(F_i) \ &\Rightarrow \alpha_i + \beta_i x = r \prod_{j \neq i} F_i(x), \\ \text{For all } x \geq 0,\ &\alpha_i + \beta_i x \geq r \prod_{j \neq i} F_i(x), \end{aligned} \tag{E-1}$$

where $\prod_{j \neq i} F_i(x)$ simply represents the probability of x being the highest performance conditioned on the performance distributions of the other managers.

In a $\boldsymbol{v}$-auction, the best reply condition is

$$\begin{aligned} x \in \text{Supp}(G_i) \ &\Rightarrow v_i \prod_{j \neq i} G_i(x) - x = u_i^*, \\ \text{For all } x \geq 0,\ &v_i \prod_{j \neq i} G_i(x) - x \leq u_i^*. \end{aligned} \tag{E-2}$$

If $\boldsymbol{G} = \boldsymbol{F}$, then, after performing the transformation specified in the lemma, we see that the equilibrium conditions for the $\boldsymbol{v}$-auction are satisfied. Thus F is a $\boldsymbol{v}$-auction equilibrium. Because the distributions of manager strategies are identical in the contest equilibrium and the corresponding auction equilibrium, it is obvious that the expected manager performance in contest equilibrium and the bidder's expected bid in the corresponding auction equilibrium are equal to each other. □

Our next lemma shows that, in a $\boldsymbol{\mu}$-contest, the marginal gain from rank competition is always lowest for manager 1, the manager with the highest capacity.

[1] Tie bids result in equal division of the rank reward between the tied managers.

Lemma E-2 *In any $\boldsymbol{\mu}$-contest equilibrium, $\beta_1 < \min_{j \neq 1} \beta_j$.*

PROOF: Suppose that this is not the case, then $\beta_1 \geq \min_{j \neq 1} \beta_j$. In the corresponding auction equilibrium specified in Lemma E-1, $v_1 \leq \max_{j \neq 1} v_j$. Let $\mathcal{I} = \{i | v_i = \max v_j\}$.

If $v_1 < \max_{j \neq 1} v_j$, then $1 \notin \mathcal{I}$. We show that this entails a contradiction by considering two cases. First suppose that $\mathcal{I}$ contains only one element, say k, then $v_k > \max_{j \neq k} v_j$. In this case, by Lemma 11 in Baye et al. (1996), F_k strictly stochastically dominates F_j, $j \neq k$. Thus, (a) $\mathbb{E}[\tilde{x}_k] > \mathbb{E}[\tilde{x}_1]$. By Lemma E-1, (b) $\mathbb{E}[\tilde{x}_k] = \mu_k$ and $\mathbb{E}[\tilde{x}_1] = \mu_1$. (a) and (b) imply that $\mu_k > \mu_i$, contradicting the definition of the $\boldsymbol{\mu}$ vector, which entails $\mu_1 > \mu_k$.

Next, suppose that $v_1 < \max_{j \neq 1} v_j$ but $\mathcal{I}$ contains more than one element, because, by hypothesis, $1 \notin \mathcal{I}$, Theorem 1 in Baye et al. (1996) shows that the 1's equilibrium auction bid is 0, which is absurd given Lemma E-1 and the fact that, by assumption, $\mu_1 > 0$.

Now consider the case where $v_1 = \max_{j \neq 1} v_j$, i.e., $1 \in \mathcal{I}$ but 1 is not the only element in $\mathcal{I}$. In this case, by Theorem 1 in Baye et al. (1996), at least two managers in $\mathcal{I}$ submit the same bid distribution. This implies that these two bidders' expected bids are the same. But, by assumption, no two components in the $\boldsymbol{\mu}$ vector are identical. And thus, given the correspondence specified in Lemma E-1, this is also impossible. □

Lemma E-3 *In any $\boldsymbol{\mu}$-contest equilibrium $\beta_2 = \beta_3 = \ldots = \beta_n$, $\alpha_2 = \alpha_3 = \ldots = \alpha_n = 0$, and $\alpha_1 > 0$.*

PROOF: Lemma E-2 and E-1 imply that, in the corresponding auction, $v_1 > \max_{j \neq 1} v_j$. Thus, we can partition the set of managers as into three sets: $\{1\}$, $\mathcal{W} = \{i | v_i = \max_{j \neq 1} v_j\}$ and $\mathcal{O} = \{i | v_i < \max_{j \neq 1} v_j\}$. Theorem 2 in Baye et al. (1996) shows that if $i \in \mathcal{O}$, then $E[\tilde{x}_i] = 0$. The correspondence established in Lemma E-1 shows that $E[\tilde{x}_i] = \mu_i > 0$; thus $\mathcal{O}$ is empty, which implies that $v_2 = v_3 \ldots = v_n$ and thus, by the correspondence, $\beta_2 = \beta_3 = \ldots = \beta_n$. Lemma E-2 combined with this result, shows that in the corresponding auction, $v_1 > v_2 = v_3 \ldots = v_n$. Theorem 2 in Baye et al. (1996) shows that in this case $u_1^* > 0$ and $u_j^* = 0$, for $j \neq 1$. Thus because, under the correspondence, $u_i^* = \frac{\alpha_i}{\beta_i}$, $\alpha_1 > 0$ and $\alpha_i = 0$ for $i \neq 1$. □

E.2 Example multi-manager contests where mixed rewards produce less risk taking than rank or bonus competitions

In this section, we present an example of the risk mitigating effect of mixed rewards a three-manager competition. In the competition, submitting the highest performance earns a rank reward, R of 1. Submitting less than the highest performance earns a rank reward, R of 0. There are three managers, S, $W1$, and $W2$ and $\mu_S > \mu_{W1} > \mu_{W2} > 0$. As shown in

Lemma E.1, the associated multipliers for the three managers must satisfy the condition that $\alpha_S > 0$, and $\alpha_{W1} = \alpha_{W2} = 0$. Since Lemma E.1 also implies that the β's of the two W managers must be the same, we represent the β's of two W managers by β_W. Lemma E-2 implies that $\beta_S < \beta_W$.

Bonus rewards

The solution in this case is apparent from Result 1, manager $i = S, W1, W2$ will submit performance equal to 0 with probability $p_i^0 = 1 - (\mu_i/\theta)$ and performance equal to θ with probability μ_i/θ.

Rank rewards

The form of equilibrium distributions for the three managers, (F_S, F_{W1}, F_{W2}) is provided by the following equation,

$$r\,F_{W1} = \begin{cases} \frac{\alpha_S + \beta_S\,x}{\sqrt{\alpha_S + \beta_S\,x_{W2}}} & x \in [0, x_{W2}), \\ \min\left[\sqrt{\alpha_S + \beta_S\,x}, 1\right] & x \geq x_{W2}; \end{cases}$$

$$r\,F_{W2} = \begin{cases} \sqrt{\alpha_S + \beta_S\,x_{W2}} & x \in [0, x_{W2}), \\ \min\left[\sqrt{\alpha_S + \beta_S\,x}, 1\right] & x \geq x_{W2}; \end{cases}$$

$$r\,F_S = \begin{cases} \frac{x\,\beta_W}{\sqrt{\alpha_S + \beta_S\,x_{W2}}} & x \in [0, x_{W2}), \\ \min\left[\frac{x\,\beta_W}{\sqrt{\alpha_S + \beta_S\,x}}, 1\right] & x \geq x_{W2}. \end{cases}$$

Because of the correspondence result, the results in Baye et al. (1996), and Lemma E.1, any rank competition equilibria must take this form for some parameters, β_S, β_W, x_{W2}, and α_S satisfying

$$0 < \beta_S < \beta_W,\; x_{W2} < \frac{1 - \alpha_S}{\beta_S}, \text{ and } \alpha_S \in (0, 1). \tag{E-3}$$

As inspection of Equation (E.2) shows that, in any equilibrium, both W managers place positive mass on 0. The probability that both W managers place mass on 0 equals α_S. S and $W1$ randomize over the interval $[0, u]$, where $u = \frac{1-\alpha_S}{\beta_S}$. $W2$ randomizes over the interval $[x_{W2}, u]$. Over the interval $[x_{W2}, u]$, the distribution functions of $W1$ and $W2$ are identical.

Mixed rewards

Now consider mixed rewards. In addition to the rank reward, the managers receive a bonus if their performance weakly exceeds the bonus threshold θ. We aim to verify an equilibrium in which S chases the bonus but $W1$ and $W2$ do not. The form of the candidate equilibrium strategies is given as follows.

$$r\,F_{W1} = \begin{cases} \frac{\alpha_S+\beta_S\,x}{\sqrt{\alpha_S+\beta_S\,x_{W2}}} & x \in [0, x_{W2}), \\ \min\left[\sqrt{\alpha_S+\beta_S\,x}, 1\right] & x \geq x_{W2}; \end{cases}$$

$$r\,F_{W2} = \begin{cases} \sqrt{\alpha_S+\beta_S\,x_{W2}} & x \in [0, x_{W2}), \\ \min\left[\sqrt{\alpha_S+\beta_S\,x}, 1\right] & x \geq x_{W2}; \end{cases}$$

$$r\,F_S = \begin{cases} \frac{x\,\beta_W}{\sqrt{\alpha_S+\beta_S\,x_{W2}}} & x \in [0, x_{W2}), \\ \min\left[\frac{x\,\beta_W}{\sqrt{\alpha_S+\beta_S\,x}}, 1-p_S^\theta\right] & x \in [x_{W2}, \theta), \\ 1 & x \geq \theta. \end{cases}$$

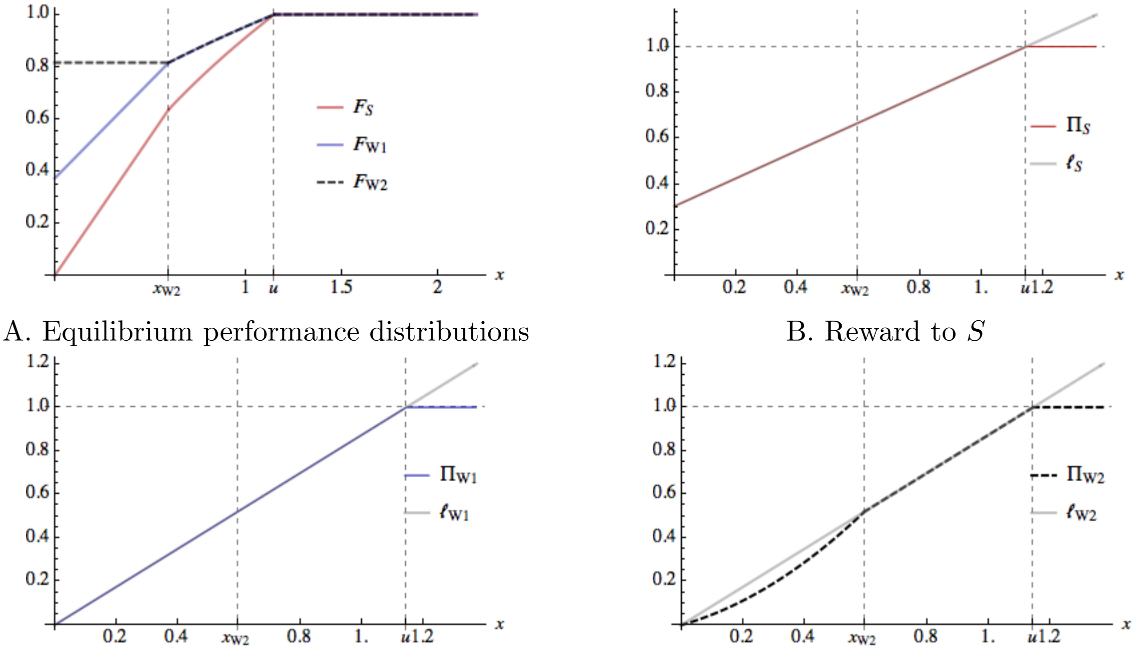
In the candidate equilibrium, the parameters also satisfy the conditions of equation (E-3) and the parameter, p_S^θ, which represents the probability that S targets the bonus, satisfies $p_S^\theta \in (0,1)$.

In order to illustrate the effects of introducing bonus compensation in a setting with more than two managers, we verify equilibria in the bonus, rank, and mixed rewards and show that qualitatively the behavior of the managers is quite similar to manager behavior in the two-manager setting modeled in the manuscript. In all three settings, the multipliers and parameters determine expected performance. However, with more than two managers, it is not possible to analytically invert the map between the parameters and expected performance. So we proceed numerically and verify the equilibria for a specific parametric case described in the following table.

Assumed capacity $\mu_S = 0.50$, $\mu_{W1} = 0.2900$, $\mu_{W2} = 0.1584$					
Assumed bonus parameters $\theta = 2.4$, $R = 1$, $B = 1.2477$					
Numerical solution					
Bonus rewards		Rank rewards		Mixed rewards	
$p_{W1}^0 = 0.8792$	$p_{W2}^0 = 0.9340$	$p_{W1}^0 = 0.3726$	$p_{W2}^0 = 0.8159$	$p_{W1}^0 = 0.2362$	$p_{W2}^0 = 0.7766$
$p_S^0 = 0.7917$	$\alpha_S = 0$	$u = 1.1446$	$\alpha_S = 0.3040$	$u = 0.9494$	$\alpha_S = 0.1834$
$\beta_S = 0.520$	$\beta_W = 0.520$	$\beta_S = 0.6081$	$\beta_W = 0.8737$	$\beta_S = 0.8601$	$\beta_W = 1.0006$
—	—	$x_{W2} = 0.5949$	—	$x_{W2} = 0.4880$	$p_S^\theta = 0.050$

Table 2: *Parametric example with multiple managers.* Note that, under mixed and rank rewards, $p_S^0 = 0$. Under bonus rewards, the parameter x_{W2} is not relevant, and under rank rewards the parameter p_S^θ is not relevant.

Plots of the manager reward functions, Π, and their support lines, ℓ, as well as the candidate equilibrium distributions, under rank and mixed rewards are provided by Figures E-1 and E-2 respectively.



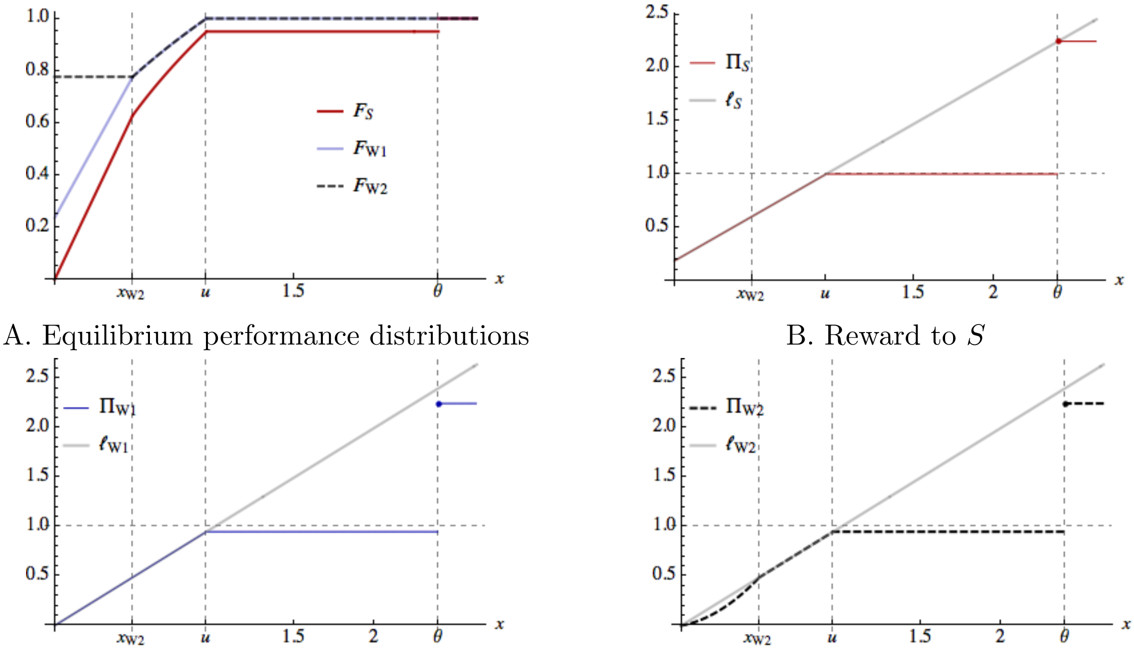
A. Equilibrium performance distributions

B. Reward to S

C. Reward to $W1$

D. Reward to $W2$

Figure E-1: *Rank rewards: Equilibrium distributions and reward functions.*



A. Equilibrium performance distributions

B. Reward to S

C. Reward to $W1$

D. Reward to $W2$

Figure E-2: *Mixed rewards: Equilibrium distributions and reward functions.*

As can be seen from the figures, the best response condition—that all performance levels in the supports of the managers' distributions lie on their support lines and no performance levels lie above their support lines—are satisfied. Numerical integration shows that the capacity constraint is satisfied with equality for all managers. Thus, the candidate equilibria are verified.

Turning to an analysis of the features of the equilibria, we see, from Table 2, that, qualitatively, equilibrium behavior is quite similar to the behavior documented in the baseline model. (a) Under mixed rewards, the probability of ruin risk taking for the two W managers, represented by p^0_{W1} and p^0_{W2}, is lower than under rank or bonus rewards. (b) The length of the subthreshold region, $(0, u]$ under mixed rewards is less than under rank rewards but performance under mixed rewards is more spread out than performance under bonus rewards. Thus, this example suggests that the qualitative effects mixed rewards are robust to extending the analysis to multiple managers.

F Implications—Identifying managerial ability

Proof: [Proof of Result 2] We only detail the slightly more difficult Eq2 case, part (i). In the Eq1 case, part (ii), the result follows from very similar arguments.

So, suppose that the configuration is Eq2. First note that the censored distribution of S performance, denoted by F_S^c equals the unconditional distribution of S performance, F_S. The censored distribution of W performance, F_W^c is given by

$$F_W^c(x) = \frac{1 - p_W^0 - p_W^h}{1 - p_W^0}\,\text{Unif}[0, u_L](x) + p_W^h\,\text{Unif}[\theta, u_H](x). \tag{F-1}$$

Consider $x \in (0, u) \cup (\theta, u_H)$, applying equation (3.1) in Lemma 1 shows that

$$\frac{F_S^{c\prime}(x)}{F_W^{c\,\prime}(x)} \;=\; \frac{1 - p_S^h - p_S^\theta}{\left(1 - p_W^0 - p_W^h\right)/\left(1 - p_W^0\right)} \;=\; \frac{p_S^h}{p_W^h/\left(1 - p_W^0\right)} \;=\; 1 - \frac{\mu_S - \mu_W}{\theta + r\,\mu_W} \;<\; 1. \tag{F-2}$$

Because $F_S^c(0) = F_W^c(0) = 0$, equation (F-2) implies that for $x \in (0, u_L]$, $F_W^c(x) > F_S^c(x)$. For $x \in [u_L, \theta)$, both F_S^c and F_W^c are constant, thus, for $x \in [u_L, \theta)$, $F_W^c(x) > F_S^c(x)$. At $x = u_H$, $F_S^c(u_H) = F_W^c(u_H) = 1$; this fact, and equation (F-2) imply that for $x \geq \theta$, $F_W^c(x) < F_S^c(x)$. Thus, F_S^c crosses F_W^c once from below.

Next note that $\mathbb{E}[X_S|X_S > 0] = \mu_S$ and $\mathbb{E}[X_W|X_W > 0] = \mu_W/(1 - p_W^0)$. The expression for p_W^0 in Lemma 1 shows that

$$\frac{\mu_W}{1 - p_W^0} - \mu_S = -\frac{(\theta - \mu_S)\,(\mu_S - \mu_W)}{(\theta - \mu_S) + \mu_W + r\,\mu_W} < 0.$$

Thus, $\mathbb{E}[X_S|X_S > 0] > \mathbb{E}[X_W|X_W > 0]$. This fact, and the fact that F_S^c crosses F_W^c once from below, imply that F_S^c second-order stochastically dominates (SSD) F_W^c. ☐

G Formal derivations of the generic properties of contest equilibria

The derivations in this appendix are novel in the sense that they have not been established in a contest game with exactly the same structure as our contest game. However, the arguments and characterizations closely track characterizations for other all pay and risk taking contest games (e.g., Siegel, 2009; Hillman and Riley, 1989). More generally, the approach taken to deriving the results—bounding the support of the payoff distribution with an envelope of affine functions that it majorizes—is equivalent to the concave envelope approach frequently used to analyze all-pay auction, risk-taking, and Bayesian persuasion games (e.g., Aumann et al., 1995; Kamenica and Gentzkow, 2011). We deal with the problem of tied performance, which generates discontinuities in the reward function, by replacing the natural reward function for the game with a more tractable reward function and then establish an equivalence between the equilibria under the natural and tractable functions. This is a very standard approach in games with discontinuous best reply correspondences (cf. Simon and Zame, 1990; Siegel, 2009).

G.1 Reward functions, best responses, ties, and support lines

Overall this section is to deal with the problem of tied performance and formally justify the support-line arguments informally developed in the paper. These results will confirm the assertions in Remarks 2 and 3 as well as confirming that the reward function used in our definition of equilibrium produces the same set of equilibria as a reward function that splits the rank reward in the event of tied performance.

G.1.1 Payoffs, best replies, and reward functions

Let –S represent W and –W represent S. Consider the following reward functions, Π_i^T and Π_i^N, both defined over $\mathbb{R}_+$:

$$\Pi_i^T(x) = F_{-i}(x-) + \tfrac{1}{2}\big(F_{-i}(x) - F_{-i}(x-)\big) + \mathbb{1}_\theta(x), \quad i = S, W, \tag{G-1}$$

$$\Pi_i^N(x) = F_{-i}(x) + \mathbb{1}_\theta(x), \quad i = S, W. \tag{G-2}$$

Π_i^T accounts for the possibility of tied performance and, in the event of tied performance, splits the rank reward of 1 equally between the two managers. As an inspection of the following derivations will show, an equal split is not essential to the arguments, any division that does not assign the entire reward to one of the managers suffices to establish the subsequent results. An equal division is used here simply to avoid introducing more notation.

In contrast, Π_i^N ignores for the possibility of tied performance and, in the event of tied performance, provides a reward of 1 to both managers. This is the reward function used in the body of the paper.

Let $\mathcal{P}^+$ be the set of all probability distribution functions supported by $[0, \infty)$ and define $\mathcal{F}_i$ by

$$\mathcal{F}_i = \left\{ F \in \mathcal{P}^+ : \int_{0-}^{\infty} x \, \mathrm{d}F(x) \le \mu_i \right\}, \quad i = S, W.$$

A best reply for manager i to F_{-i} under Π^k, $k = T, N$ is a probability distribution F^* that satisfies

$$\int_{0-}^{\infty} \Pi_i^k(x)\,\mathrm{d}F^*(x) = \sup\left\{\int_{0-}^{\infty} \Pi_i^k(x)\,\mathrm{d}F(x) : F \in \mathcal{F}_i\right\}, \quad k = T, N,\ i = S, W.$$

Both Π_i^T and Π_i^N are bounded, nondecreasing functions defined over $\mathbb{R}_+$ that only differ with respect to how they treat tied performance. Tied performance occurs with positive probability if and only if both managers choose discontinuous performance distributions that both put positive mass on some performance level. To formalize this notion let

$$\mathcal{M}_i = \{x \geq 0 : F_i(x) - F_i(x-) > 0\}, \quad i = S, W. \tag{G-3}$$

Note that because distributions are non decreasing, $\mathcal{M}_i$ is at most countable. We will say that tied performance does not occur if $\mathcal{M}_S \cap \mathcal{M}_W = \varnothing$. When ties do not occur, $\Pi^T = \Pi^N$.

Π^N is right-continuous, and this implies, because it is also non-decreasing, that it is upper semicontinuous. This implies that the map

$$F \hookrightarrow \int_{0-}^{\infty} \Pi^N(x)\,\mathrm{d}F(x)$$

is upper semicontinuous (Ash, 1972, Theorem 4.5.1.b). Thus, whenever the payoff from sequence of distribution converges to the supremum of a manager's payoff, and the sequence of distributions converges to a distribution function F_o, F_o is a best reply.

In contrast, in general, Π^T is not upper semicontinuous. Thus, the limit of a convergent sequences of distributions producing payoffs that converge to the supremum, need not be a best reply. For this reason, Π^T is not a very convenient reward function. However, it does accurately reflect the fact that the total rank reward is constant, while Π^N does not.

Fortunately, as we will show below, the set of Nash equilibria given Π^T equals the set of Nash equilibria under Π^N which satisfy the condition of no-tied performance. This equivalence results simply because, under Π^T, managers will never choose in equilibrium to submit performance distributions that result in performance ties with positive probability. This general approach is used in Siegel (2009) to apply Simon and Zame (1990) existence result for games with discontinuous best reply correspondences to all-pay auctions.

G.1.2 Support lines and optimal performance distributions

To establish this equivalence, and to characterize the general properties of equilibrium performance distributions, we require a simple means of characterizing best replies. This desideratum will be supplied by a support line characterization of best replies developed in this section.

Because, in this section, we are only concerned with characterizing the properties of optimal performance distribution for an individual manager under a given reward function, we will simplify notation by suppressing the subscript representing agent type. Because the results in this section hold for both Π^N and Π^T we will simply represent the contest reward function with Π. The manager's problem is to maximize her payoff under the contest reward function,

Π, over distribution functions in $\mathcal{P}^+$ subject to the capacity constraint, let ν^* represent the supremum of this problem, i.e.,

$$\nu^* = \sup_{F\in\mathcal{P}^+} \left\{ \int_{0-}^{\infty} \Pi(x)\,\mathrm{d}F(x) : \int_{0-}^{\infty} x\,\mathrm{d}F(x) - \mu \le 0 \right\}. \tag{G-4}$$

Because, For all $x \ge 0$, $\Pi(x) \in [0, 1+r]$, $\nu^* < \infty$. Define the associated Lagrange function, $\mathcal{L} : \mathcal{P}^+ \times [0,\infty) \to \mathbb{R}$:

$$\mathcal{L}(F,\lambda) = \int_{0-}^{\infty} (\Pi(x) - \lambda\,x)\ \mathrm{d}F(x) + \lambda\,\mu. \tag{G-5}$$

The objective function,

$$\mathrm{d}F \hookrightarrow \int_{0-}^{\infty} \Pi(x)\,\mathrm{d}F(x)$$

is linear and hence concave, the constraint set is convex, and clearly, there exists F such that the constraint is strictly satisfied. Thus, there exists a multiplier, $\lambda^* \ge 0$, such that

$$\nu^* = \sup_{F\in\mathcal{P}^+} L(F,\lambda^*), \tag{G-6}$$

and, if the supremum in (G-4) is attained at F^*, then F^* attains the supremum in equation (G-6) (Chapt 1., Theorem 1 Luenberger, 1969). Let

$$\beta = \lambda^*, \qquad \text{and } \alpha = \sup_{x\ge 0} \Pi(x) - \beta\,x. \tag{G-7}$$

Note that because $\Pi(0) \ge 0$, $\alpha \ge 0$.

Thus, if F^* is an optimal policy, using equations (G-5) and (G-7), we can express the Lagrange function, evaluated at F^* and λ^*, in terms of α and β as follows:

$$\nu^* = \sup_{F\in\mathcal{P}^+} \mathcal{L}(F,\lambda^*) = \mathcal{L}(F^*,\lambda^*) = \int_{0-}^{\infty} \Big(\Pi(x) - (\alpha + \beta\,x)\Big)\,\mathrm{d}F^*(x) + \alpha + \beta\,\mu. \tag{G-8}$$

Inspection of equation (G-8) implies that the optimal performance distribution, F^*, satisfies

$$\begin{aligned} \mathrm{Supp}(F^*) &\subseteq \{x \ge 0 : \Pi(x) - (\alpha + \beta\,x) = 0\}, \\ &\forall x \ge 0,\ \Pi^*(x) \le \alpha + \beta\,x, \\ 0 &= \int_{0-}^{\infty} \Big(\Pi(x) - (\alpha + \beta\,x)\Big)\,\mathrm{d}F^*(x). \end{aligned} \tag{G-9}$$

Clearly $\beta > 0$. If $\beta = 0$ then $\alpha = 1 + r$, which implies, by (G-8) and (G-9), that $\nu^* = 1 + r$, which is impossible by our assumption that $\mu < \theta$, and thus the bonus cannot be captured with probability 1.

Lemma G-1 (**Multipliers and support lines**) *If* $\Pi = \Pi^T$ *or* Π^N *and* F^* *is a best response to* Π*, there exists* $\alpha \ge 0$ *and* $\beta > 0$ *and support line,* $\ell(x) = \alpha + \beta\,x$*, such that* F^* *satisfies*

$$\mathrm{Supp}(F^*) \subseteq \{x \ge 0 : \Pi(x) = \ell(x)\}, \tag{G-10}$$

$$\forall x \ge 0,\ \Pi(x) \le \ell(x). \tag{G-11}$$

Lemma G-1 confirms Remark 3. Note that Lemma G-1 establishes necessary conditions for optimal performance distributions under both Π^T and Π^N. These conditions do not speak to the question of whether an optimal performance distribution exists, i.e., whether the supremum of the managers' optimization problems is attained. In this respect, Π^T and Π^N can differ substantially, as pointed out earlier.

G.1.3 Ties

The next result confirms Remark 1 by showing that, under Π^T, best responses never produce tied performance. Intuitively, this is obvious, at a tie point, a manager can divert infinitesimal to slightly increasing performance at the tie point and to “just top” her rival, breaking the tie and increasing her payoff by a non infinitesimal amount.

Lemma G-2 *If F_i^* is best response to Π^T, then $\mathcal{M}_{-i} \cap \mathrm{Supp}_i = \varnothing$, i.e., a best response by i to $-i$ never produces tied performance.*

Proof: Suppose not. Then there exists $x_o \geq 0$ to which both managers assign positive probability mass. In the event of a tie, the rank reward is divided between the two managers, with each receiving a rank-based reward of 1/2. Let $\{x_n\}$ be a decreasing sequence converging to x_o.

Consider a manager's, say S, reward function. The rank based-reward to W if W plays F_W is

$$\Pi_S^T(x_o) = \mathbb{P}[X_W < x_o] + \tfrac{1}{2}\,\mathbb{P}[X_W = x_o], \quad \mathbb{P}[X_W = x_o] > 0. \tag{G-12}$$

The rank based reward to S from x_n equals

$$\Pi_S^T(x_n) = \mathbb{P}[X_W < x_n] + \tfrac{1}{2}\,\mathbb{P}[X_W = x_n] \geq \mathbb{P}[X_W \leq x_o] = \mathbb{P}[X_W < x_o] + \mathbb{P}[X_W = x_o]. \tag{G-13}$$

The bonus reward at x_n is no less than the bonus reward at x_o. Thus,

$$\Pi_S^T(x_n) \geq \Pi_S^T(x_o) + \tfrac{1}{2}\,\mathbb{P}[X_W = x_o]. \tag{G-14}$$

Because $x_o \in \mathrm{Supp}_S$ by hypothesis, condition (G-10) implies that $\ell_S(x_o) = \Pi_S(x_o)$. In order for condition (G-11) to be satisfied, it must be the case that

$$\forall n \in \mathbb{N}, \quad \ell_S(x_n) \geq \Pi_S^T(x_n). \tag{G-15}$$

Equations (G-14) and (G-15) imply that

$$\forall n \in \mathbb{N}, \quad \ell_S(x_n) \geq \Pi_S^T(x_o) + \tfrac{1}{2}\,\mathbb{P}[X_W = x_o].$$

Thus

$$\Pi_S^T(x_o) = \ell_S(x_o) = \lim_{n\to\infty} \ell_S(x_n) \geq \Pi_S^T(x_o) + \tfrac{1}{2}\,\mathbb{P}[X_W = x_o] > \Pi_S^T(x_o).$$

This contradiction establishes the lemma. □

Lemma G-2 provides an equivalence relation between equilibria under Π^T and Π^N which rationalizes the reward function used in the body of the paper (equation (2.1)).

Lemma G-3 *The following statements are equivalent:*

(i) For $i = S, W$, F_i is a best response to F_{-i} under Π_i^T,
(ii) For $i = S, W$, F_i is a best response to F_{-i} under Π_i^N and $\mathcal{M}_S \cap \mathcal{M}_W = \varnothing$.

PROOF: (i) $\Rightarrow$ (ii): If (i) holds, then Lemma G-2 implies that $\mathcal{M}_S \cap \mathcal{M}_W = \varnothing$, which implies that $\Pi_i^T = \Pi_i^N$ and thus, (ii) holds.

(ii) $\Rightarrow$ (i): If (ii) holds then $\Pi_i^T = \Pi_i^N$, and thus (i) holds. $\square$

This result formally shows that the payoff function used in the body of the paper, which in the body of the paper is simply called Π and in this appendix, thus far, has been termed Π^N, characterizes equilibrium behavior even when the game specifies a division of the rank reward in the event of ties.

G.2 Properties of equilibrium performance strategies

Henceforth, making a slight abuse of notation we represent Π_i^N, $i = S, W$, *simply by* Π_i. Note that this definition coincides with the definition of Π_i in the main body of the paper.

Lemma G-4 *If* (F_W, F_S) *are equilibrium performance distributions,*

(i)

$$(0, \theta) \cap \mathrm{Supp}_S = (0, \theta) \cap \mathrm{Supp}_W,$$

(ii)

$$(\theta, \infty) \cap \mathrm{Supp}_S = (\theta, \infty) \cap \mathrm{Supp}_W.$$

PROOF: We prove (i), the proof of (ii) is omitted because it is virtually identical. Suppose to obtain a contradiction, that there exists x_o such that $x_o \in (0, \theta) \cap \mathrm{Supp}_W$ but $x_o \notin (0, \theta) \cap \mathrm{Supp}_S$ (the argument with the roles of the managers reversed is identical up to transpositions of the type names).

Because, by definition, Supp_S is closed, there exists an open neighborhood N of x_o in $(0, \theta)$, such that, for all $x \in N$, $x \notin \mathrm{Supp}_S$. Over $(0, \theta)$, $\Pi_W = F_S$ and thus because F_S is constant on N, Π_W is constant on N.

By hypothesis, $x_o \in N \cap \mathrm{Supp}_W$, thus by condition (G-10), $\ell_W(x_o) = \Pi_W(x_o)$. Because ℓ_W is increasing, for $x \in N$ and $x < x_o$, $\ell_W(x) < \ell_W(x_o)$ and, because Π_W is constant over N, $\Pi_W(x) = \Pi_W(x_o)$. Thus, for $x \in N$ and $x < x_o$, $\ell(x)_W < \Pi_W(x)$, contradicting condition (G-11). $\square$

Lemma G-4 confirms Remark 2.(a) in the main body of the paper. The next result shows that, in all equilibria, both managers assign some probability weight to the subthreshold region.

Lemma G-5 *If* (F_W, F_S) *are equilibrium performance distributions,*

$$(0, \theta) \cap \mathrm{Supp}_i \neq \varnothing, \quad i = S, W. \tag{G-16}$$

PROOF: If $(0,\theta)\cap\text{Supp}_i = \varnothing$, then, by Lemma G-4, $(0,\theta)\cap\text{Supp}_j = \varnothing$, $j \neq i$. Performance distributions that are supported by $[\theta,\infty)$ are inconsistent with the capacity constraint given the assumption that $\mu_i < \theta$, $i = S, W$. Thus, both S and W must place positive mass on 0, i.e., the probability of tied performance must be positive. But this is impossible by Lemma G-2. $\square$

Lemma G-5 will now be used to derive the next lemma, Lemma G-6, which confirms Remark 2.(d).

Lemma G-6 *If (F_W, F_S) are equilibrium performance distributions,*

(i)

$$0 \in \text{Supp}_i, \quad i = S, W.$$

(ii)

$$\textit{If}\, \text{Supp}_i \cap [\theta,\infty) \neq \varnothing,\ \theta \in \text{Supp}_i, \quad i = S, W.$$

PROOF: Again we prove only (i), because the proof of (ii) is virtually identical. Suppose, to obtain a contradiction, that 0 is not in the support of one of the manager's performance distributions, say S. Then there would exist a neighborhood of 0, open in $[0,\infty)$ such that for all $x \in N$, $x \notin \text{Supp}_S$. This implies that on N, F_S is constant. Let $x_o = \min\{x \geq 0 : x \in \text{Supp}_S\}$. By Lemma G-5, $x_o < \theta$. For $x \in [0, x_o)$, F_S is constant and thus Π_W is constant. Thus, for all $x \in [0, x_o)$, $\Pi_W(x) = \Pi(0)$, which implies that $\Pi(x_o-) = \Pi(0)$. If Π_W is continuous at x_o (i.e., F_S is continuous at x_o) then $\Pi_W(x_o) = \Pi_W(0)$. Condition (G-11) implies that $\ell_W(0) \geq \Pi_W(0)$. Because ℓ_W is increasing, and $\Pi(x_o-) = \Pi(0)$, $\ell(x_o) > \Pi_W(x_o)$, which, by condition (G-10), implies that $x_o \notin \text{Supp}_W$. By Lemma G-4 this implies that $x_o \notin \text{Supp}_S$, contradicting the definition of x_o.

Thus, F_S must jump at x_o. This implies by Lemma G-2, that F_W does not jump at x_o. On N, F_W is constant, and does not jump at x_o. Hence, $\Pi_S(x_o) = \Pi_S(0)$. Because, by construction, $x_o \in \text{Supp}_S$, $\ell_S(x_o) = \Pi_S(x_o)$ by condition (G-10). Because Π_S is constant on N and ℓ_S is increasing, for $x \in [0, x_o)$, $\ell_S(x) < \Pi_S(x)$, contradicting condition (G-11). $\square$

In addition to confirming Remark 2.(d), Lemma G-6 shows that when the superthreshold region is not empty, its greatest lower bound always equals the bonus threshold. The following lemma shows that the sub- and superthreshold regions are in fact intervals confirming Remark 2.(b).

Lemma G-7 *If (F_W, F_S) are equilibrium performance distributions,*

(i)

$$\text{Supp}_i \cap [0,\theta) \textit{ is connected.}$$

(ii)

$$\textit{If}\, \text{Supp}_i \cap [\theta,\infty) \neq \varnothing \Rightarrow \text{Supp}_i \cap [\theta,\infty) \textit{ is connected.}$$

PROOF: We shall prove (i). The proof of (ii) is virtually identical save for adding the bonus compensation reward to the contest reward function. To obtain a contradiction suppose that $\mathrm{Supp}_i \cap [0, \theta)$ is not connected. Without loss of generality, suppose that $i = S$. For $\tau, \nu > 0$, let

$$G = \bigcup_{\tau,\nu>0} \{(x_o - \tau, x_o + \nu) : (x_o - \tau, x_o + \nu) \cap \mathrm{Supp}_S = \varnothing\}. \tag{G-17}$$

Let

$$\underline{x} = \inf G, \quad \bar{x} = \sup G.$$

Then $\underline{x}, \bar{x} \in \mathrm{Supp}_S$ and thus, by Lemma G-4 and G-6, $\underline{x}, \bar{x} \in \mathrm{Supp}_W$. Thus, condition (G-10) implies that $\ell_S(\underline{x}) = \Pi_S(\underline{x})$, $\ell_S(\bar{x}) = \Pi_S(\bar{x})$, $\ell_W(\underline{x}) = \Pi_W(\underline{x})$, and $\ell_W(\bar{x}) = \Pi_W(\bar{x})$. Because, for $x < \theta$, $\Pi_S = r\, F_W$ and $\Pi_W = r\, F_S$ and because G does not meet the supports of S and W, Π_S and Π_W are constant on G. Thus because ℓ_S and ℓ_W are increasing, it must be the case that both F_S and F_W jump up at $\bar{x}$, this implies tied performance with positive probability at $\bar{x}$, contradicting Lemma G-2. □

The next two results, Lemma G-8 and Lemma G-9, are fairly obvious technical results that will be used to establish our final characterization, continuity, in Lemma G-10.

Lemma G-8 *If (F_W, F_S) are equilibrium performance distributions,*

$$\sup\{\mathrm{Supp}_i \cap [0, \theta)\} < \theta.$$

PROOF: $\Pi_i(\theta) > \Pi_i(\theta-) + 1$. Condition (G-11) implies that $\ell_i(\theta) \geq \Pi_i(\theta)$, $\ell_i(\theta) = \ell_i(\theta-)$, thus $\ell_i(\theta) > \Pi_i(\theta-)$. Because ℓ is continuous, this implies that for all x is a sufficiently small lower neighborhood of θ, $\ell_i(\theta) > \Pi_i(x)$. Thus by condition (G-10), such x are not in Supp_i. □

Lemma G-9 *If (F_W, F_S) are equilibrium performance distributions,* $\sup(\mathrm{Supp}_i) < \infty$, *$i = S, W$.*

PROOF: We establish this result for S, the proof for W is identical save for transpositions of S and W. For $x \geq \theta$, $\Pi_S(x) = 1 + r\, F_S(x) \leq 1 + r$ and $\lim_{x\to\infty} \ell_S(x) = \infty$. So for x sufficiently large, $\ell_S > \Pi_S$, implying, by condition (G-11), that for x sufficiently large, $x \notin \mathrm{Supp}_S$. □

Finally, we confirm Remark 2.(c).

Lemma G-10 *If (F_W, F_S) are equilibrium performance distributions, then F_S and F_W are continuous except perhaps at 0 and θ.*

PROOF: Suppose that one of the distributions, say F_S, has a jump at $x_o \neq \theta$ or 0. Suppose that $x_o < \theta$. The proof when $x_o > \theta$ is the same except that the bonus reward is added to the payoffs. In this case, obviously $x_o \in \mathrm{Supp}_S$ which implies, by Lemma G-4, that $x_o \in \mathrm{Supp}_W$. For $x < x_o < \theta$, $\Pi_W = r\, F_S$, thus, by condition (G-10), $\ell_W(x_o) = \Pi_W(x_o)$. Lemma G-6 shows that $0 \in \mathrm{Supp}_W$, so, again by condition (G-10), $\ell_W(0) = \Pi_W(0)$. Because Π_W jumps up at x_o and ℓ is continuous, for a sufficiently small lower neighborhood of x_o, $\Pi_W(x) < \ell_W(x)$. By condition (G-10) this implies that points in this neighborhood are not in Supp_W. Thus, there exists $0 < x' < x_o$ such that $x' \notin \mathrm{Supp}_W$ but $0 \in \mathrm{Supp}_W$. $x_o \in \mathrm{Supp}_W$, i.e., Supp_W is not connected, contradicting Lemma G-7. □

This result confirms Remark 2.(c) in the body of the paper. The following lemma simply summarizes the implications of the previous lemmas for equilibrium manager reward functions.

Lemma G-11 *If (F_W, F_S) are equilibrium performance distributions, then there exist constants, α_S, β_S, α_W, β_W, u_H, u_L, such that (a) $\beta_S, \beta_W > 0$ and $\alpha_S, \alpha_W \geq 0$, (b) $u_L \in (0, \theta)$, (c) $u_H \in [\theta, \infty)$, and*

(i) If $x \in [0, \theta)$,

$$\Pi_i(x) = \begin{cases} \alpha_i + \beta_i\, x & x \in [0, u_L], \\ \Pi_i(u_L) & x \in (u_L, \theta). \end{cases}$$

(ii) If $x \in [\theta, \infty)$ and $u_H = \theta$,

$$\theta \in \mathrm{Supp}_S \Leftrightarrow \theta \notin \mathrm{Supp}_W,$$
$$\theta \in \mathrm{Supp}_i \Rightarrow \alpha_i + \beta_i\, \theta = 1 + r, \quad \theta \notin \mathrm{Supp}_i \Rightarrow \alpha_i + \beta_i\, \theta \leq 1 + r, \text{ and}$$
$$\Pi_i(x) = 1 + r, \quad i = S, W.$$

(iii) If $x \in [\theta, \infty)$ and $u_H > \theta$,

$$\Pi_i(x) = \begin{cases} \alpha_i + \beta_i\, x & x \in [\theta, u_H], \\ 1 + r & x > u_H. \end{cases}$$

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