The Drivers and Inhibitors of Factor Investing

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Abstract

I model the equilibrium asset allocations when households can invest directly, search for factor (smart-beta and ETFs) investments or fundamental (stock-picking) investments. Managers endogenously choose to specialize in factor or fundamental information given the equilibrium fee structure. Fundamental managers can opt to be opportunistic "closet indexers." I show that wealth *inequality* increases demand for factor investing: fundamental investing attracts the wealthiest households, who are more willing to detect closet indexing. Fundamental managers have to compete more aggressively through information acquisition, which lowers their excess returns and thus delegation fees. The reduced fundamental fees force factor managers to lower their fees, making factor investing even more attractive. However, the equilibrium fraction of capital allocated to factor investing can never reach 100 percent: the ceiling is determined by the endogenous level of opportunism in the fundamental investment industry.

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1 Introduction

In August 2019, the net asset value of U.S. indexed funds reached \$4.27 trillion, surpassing for the first time ever the money managed by their stock-picking rivals.[1] A large proportion of this increase in passive allocation is explained by factor investing: over \$1.9 trillion is currently invested in factor funds, including index-based vehicles, rules-based ETFs, and smart-beta products; and this number is expected to swell to \$3.4 trillion in 2022.[2]

Factor investing allows households to buy market beta and hold broadly diversified portfolios with minimal delegation fees. In contrast, traditional active funds exploit differences between the market price and the fundamental value of individual stocks through rigorous security selection. As factor investors partly free-ride on the price discovery efforts of fundamental investors, the rise of factor investing might distort asset prices and affect price efficiency.[3] This raises a number of questions: What causes the increase in factor investing? If the rise is problematic, is there an upper bound of the fraction of capital that will ultimately be allocated to factor investing?

In this paper, I address the questions through a rational expectations equilibrium (REE) model, which jointly determines the (i) equilibrium asset prices, (ii) households' allocation between factor, fundamental, and direct investing, (iii) managers' selection into different roles, and (iv) delegation fees. Investment in acquiring fundamental information is costly and unobservable. Thus active managers can opt to become "closet" indexers, who opportunistically market themselves as fundamentally informed, despite holding index-like portfolios.[4] Analysis of the equilibrium reveals that there exists an implicit *ceiling* for factor investing: as factor investing increases, more and more of the market-level information is impounded into asset prices. This effect makes the return differential between fundamental investing and factor investing larger, and thus ensures that factor investing will never completely dominate fundamental investing. The upper bound of factor investing is lower when it is harder for managers to obtain fundamental information, or when it is easier for households to detect closet indexing.

A novel feature of this work is to incorporate elements of household finance into the standard REE model. I introduce heterogeneity among households in the form of wealth dispersion and examine how this affects capital allocation in the fund industry.[5] I find that *wealth inequality* fuels demand

[1] 'Index Funds Are the New Kings of Wall Street,' the Wall Street Journal, September 2019. Figure A-1 in the appendix summarizes the facts.

[2] 'Can Factor Investing Kill Off the Hedge Fund?' Financial Times, July 2018; 'Hybrid Funds Smooth Path Between Active and Passive Strategies,' Financial Times, November 2019.

[3] Empirical evidence can be found in Coles, Heath, and Ringgenberg (2018) and Israeli, Lee, and Sridharan (2017).

[4] The terminology "closet" fund first appears in Oppel (1999), and is widely used in the later empirical studies, e.g., Petajisto (2013) and Cremers et al. (2016). Cremers and Petajisto (2009) find that internationally, about 20% of the mutual fund assets are managed by closet funds.

[5] None of the existing literature has taken into account how household inequality is linked to capital allocation in the fund industry, though there exist some studies examining the effect of household heterogeneity on asset prices (cf. Peress, 2003; Gomes and Michaelides, 2007).

for factor investing and leads to compression of delegation fees. Fundamental investing attracts the wealthiest households, who are most willing to detect closet indexing. Thus active managers have to increase information acquisition, curtailing the marginal gain from information production. This effect brings market prices closer to their fundamentals, endogenously generating decreasing returns to scale of the fundamental investment industry and forcing active funds to lower fees to retain capital. Factor funds, accordingly, charge less and gain popularity.

To derive these ideas, I extend Grossman and Stiglitz (1980) model to a multi-asset general equilibrium. The market is not completely efficient and is subject to exogenous and noisy supply of assets. Managers are incentivized and endogenously choose to specialize in factor or fundamental information. Households can detect the true skill of active managers through a costly search, or invest based on their prior information. Compared to factor investing, fundamental investing is associated with higher active fees, and an implicit cost arising from opportunism. The delegation fees are endogenously set. The model shows that factor investing increases competitive pressure, forcing active funds to lower fees, thereby providing theoretical justification for the finding of fee compression in Cremers et al. (2016).

The equilibrium capital allocation of households depends on security market characteristics, risk preferences, information production costs, and overall market efficiency. Generally, factor investing is more popular than fundamental investing due to its efficient cost and the endogenous opportunism in the active industry. However, when the costs of obtaining fundamental information are sufficiently low, or when the gains from trading on idiosyncratic risks are sufficiently large, factor investing will be driven out of the market. Above a certain threshold, factor strategies suffer a severe reduction in the diversification benefit, as the constituents in the index comove a lot and index price becomes highly informative, driving households to fundamental investing.

To see if capital allocation and other equilibrium characteristics persist in a more realistic context, I introduce household heterogeneity to the baseline model in two dimensions: wealth inequality and heterogeneous cognitive ability of finding and vetting a manager. Household heterogeneity has the following two implications: (i) while all three investment vehicles coexist, at the individual level households have different preferences with regard to their capital allocation; and (ii) the allocation strategies from individuals with different wealth positions, thus, contribute differently to funds' inflows. First, heterogeneity yields insights into the likely composition of fundamental investing households: those with higher wealth endowments or more insight into the quality of funds, favor fundamental investing. In the case of cognitive ability, more able households can more easily discern fund manager opportunism; in the case of wealth, wealthier households exhibit less absolute risk aversion at the margin.

Second, the model predicts that higher inequality fuels demand for factor investing, mainly at the expense of direct investing. The wealth effect is more prominent than the cognitive ability effect. Hence, household inequality serves as one possible explanation for rising indexation, as both time trends can be observed simultaneously in countries such as the U.S.[6] Increased inequality concentrates wealth in the hands of less risk-averse wealthy households, who favor fundamental investing and have relatively low searching cost to detect closet indexing. This increases the attractiveness of managers becoming skilled, leading to higher inside competition. Thus the outperformance of fundamental strategy shrinks, making households less interested in fundamental investing. Moreover, the reduced fees earned by active funds force factor funds to lower their fees, making factor investing even more attractive compared to direct investment.

Third, in a heterogeneous equilibrium, market competition can never completely eliminate closet indexers. For instance, when fundamental information production is rather costly, active managers charge a high premium and are required to generate large excess return to justify this charge. However, the less wealthy or less able households can never be incentivized to fully invest with skilled managers, limiting the gains from fund inflows through skill acquisition. In contrast, in the homogeneous world, households are *ex-ante* identical and can exert effort to learn about managerial abilities. Facing the high cost of active delegation, *ceteris paribus*, households are more likely to become informed to avoid selection error. As a consequence, active funds have to be truly skilled to secure inflows, and closet indexers are wiped out.

This paper provides a new lens for understanding the drivers of the rise in factor investing. I find that increased factor investing can be attributed to the following factors: (i) A falling but not insignificant searching cost for households: widespread access to information online makes it easier for households to learn about managerial abilities through assessing managers' investment principles, trading infrastructure, etc. A decrease in searching cost lowers the excess returns generated by fundamental investors, and thus spurs more factor strategies.[7] (ii) A falling but not degenerate fundamental information cost for managers: this is highly possible due to more transparency in the corporate information environment and more developed techniques like data mining. Lower information production cost forces both funds to charge less, making professional delegation more attractive. (iii) An intermediate level of risk aversion: households are neither so afraid of risk that they prefer riskless assets, nor so risk-inclined that they ignore the volatility of undiversified portfolios.[8] (iv) Higher idiosyncratic risk: if the idiosyncratic factors are volatile or their supply is highly uncertain, risk-averse households seek factor investing as a substitute. (v) Increased inequality: as explained above, more efficient individual prices and fiercer competition within the active industry encourage factor investing. These findings help understand a number of empirical observations. The comparative statics have further implications on return volatility, co-movement, and price efficiency.

The paper contributes to an expanding and diverse body of literature. There exists massive literat-

[6]See a time-series summary of wealth inequality in Figure A-2, which lines up with the time trend in Figure A-1.

[7]In the context of my model, I find that a sufficiently small searching cost engenders a significantly low level of opportunism, and thus leads households to shift towards fundamental investing and away from factor investing. However, it is highly unlikely that the searching costs are negligible in a real-world context.

[8]This non-monotonic effect of risk appetite contradicts the argument in Glasserman and Mamaysky (2018).

ure modeling delegated portfolio management. Some introduce household behavioral biases, such as what is termed "trust" in Gennaioli, Shleifer, and Vishny (2015). Others take rational expectations assumption built on the Grossman and Stiglitz (1980) framework. Among these REE studies, Gârleanu and Pedersen (2018) model the investors searching for active managers. However, their model is restricted to single-asset setup. Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016) take the attention allocation exogenous and in their framework all informed agents ultimately form identical strategy. In closely-related contemporaneous work, Glasserman and Mamaysky (2018) and Gârleanu and Pedersen (2019) endogenize the information choice of agents and prove Samuelson's dictum. This work is in contrast with theirs, in that it delivers distinct prediction on the drivers of factor investing, and explicitly models the opportunistic closet funds. None of these studies address the link between household heterogeneity and their delegation choices.

The other way of deriving the equilibrium is through the Kyle (1985) framework. For instance, Cong and Xu (2016) examine the rising factor investing from the security design perspective, on how it affects trading behaviors and asset prices. Bhattacharya and O'Hara (2018) borrow a similar setting and document that ETF activities introduce fragility and alter the information efficiency of underlying markets via herding. But they both introduce indexing activity as exogenous, which, in my model, instead, is one of the pertinent features of asset market equilibrium.

Empirically, Sorensen, Miller, and Samak (1998) discuss the allocation trade-off between active and index funds, with the optimal strategy of an asset owner largely depending on the stock-picking skills of active funds. In fact, even though on average, active industry does not outperform indexation, a bunch of literature documents that a modest amount active managers possess "skills" to generate excess return, including both stock-picking and market-timing.[9] The allocation between different investment vehicles may have further implications on fund structure and fees, as what is emphasized in this paper. Vayanos and Woolley (2016) raise the concern that adopting index-based strategy can circumvent the agency problem. International evidence has been found by Cremers et al. (2016), that increased indexing forces the active rivals to lower fees.

To the best of my knowledge, this paper is the first to combine the standard REE model with household finance, and yield clear prediction of household heterogeneity affecting delegation preference and asset market. Wealth inequality and asymmetric cognitive ability among households, nevertheless, have long been crucial social-economic topics. High wealth inequality implies low future stock returns, as found by Gomez et al. (2019). Concerning cognitive ability, Grinblatt, Keloharju, and Linnainmaa (2011) examine its relation with stock market participation. Benjamin, Brown, and Shapiro (2013) conduct a laboratory study which links cognitive capability to risk preference. Some economic literature examines similar questions in a different background, including Proto,

[9] For reference see Kosowski et al. (2006); Kacperczyk and Seru (2007); Mamaysky, Spiegel, and Zhang (2008); Cuthbertson, Nitzsche, and O'Sullivan (2008); Cremers and Petajisto (2009); Kacperczyk, Nieuwerburgh, and Veldkamp (2014); Berk and Van Binsbergen (2015); Choi, Kahraman, and Mukherjee (2016); Pástor, Stambaugh, and Taylor (2017).

Rustichini, and Sofianos (2019) that design an experiment on how heterogeneous intelligence level affects outcomes of groups.

The empirical literature towards the effect of indexing on asset price efficiency is growing, though a bit mixed. Israeli, Lee, and Sridharan (2017) find that over the long run ETF activities deteriorate price efficiency. However, Glosten, Nallareddy, and Zou (2016) argue that ETFs improve especially price efficiency for stocks with weak information environments or low liquidity. The amplification of factor investing also affects return volatility and co-movement of assets.[10]

The remainder of the paper is structured as follows. Section 2 describes the economic setup. I derive the asset market equilibria in Section 3. Section 4 introduces heterogeneity of households to the standard REE model. Section 5 summarizes the possible drivers of rising factor investing, and uses them to interpret other pertinent features of asset market equilibria. Section 6 concludes. All proofs are in the appendix.

2 Economic Setup

2.1 Assets

Assume there are $l+1$ underlying assets in the market. One is the riskless asset with gross return normalized to be 1. The other l risky assets deliver the payoffs $\boldsymbol{v} = (v_1, v_2, ..., v_l)^\intercal$, where

$$v_i = \beta_i S + \mathscr{E}_i, \quad i = 1,2,...,l. \tag{2.1}$$

Intuitively, each asset can be decomposed to include a *factor loading*, S (measuring systematic risk shared by all publicly traded assets), and an *idiosyncratic loading*, $\mathscr{E}_i$ (measuring, for instance, how a firm-specific shock affects stock value). S and $\mathscr{E}_i$ are independent and normally distributed, with $S \overset{d}{\sim} \mathscr{N}(\bar{s}, \sigma_s^2)$ and $\mathscr{E}_i \overset{d}{\sim} \mathscr{N}(0, \sigma_e^2)$, $i \in \{1,2,...,l\}$. β_i captures asset i's exposure to the aggregate market. Any portfolio held by an investor can be decomposed to an l-dimensional vector, representing the holding of the risky assets, denoted by $\boldsymbol{x} \in \mathbb{R}^l$. The holding of riskless asset is thus automatically determined.

The index portfolio is a bundle of underlying assets, with a composition of $\boldsymbol{x_I} = (b_1, b_2, ..., b_l)^\intercal$, where $\sum_i^l b_i = 1$. Its payoff is $\sum_i^l b_i v_i = S$, i.e., corresponding to the market portfolio. This implies that (i) $\sum_i^l b_i \beta_i = 1$ and (ii) $\sum_i^l b_i \mathscr{E}_i = 0$. Condition (ii) can be guaranteed by the assumption that,

[10]There exist empirical discussions from academia (e.g., Glosten, Nallareddy, and Zou, 2016; Israeli, Lee, and Sridharan, 2017; Da and Shive, 2018; Ben-David, Franzoni, and Moussawi, 2018; Coles, Heath, and Ringgenberg, 2018) as well as from regulation and law authorities (e.g., Sushko and Turner, 2018; Anadu et al., 2018; Rock and Kahan, 2018; James et al., 2019).

within the index portfolio, idiosyncratic risks are averaged out, i.e.,

$$Cov(\mathscr{E}_i, S) = Cov(\mathscr{E}_i, \sum_{k=1}^{l} b_k \mathscr{E}_k) = 0, \quad \forall i \in \{1,2,...,l\}.$$

These l equations imply a feasible $l \times l$ correlation matrix of the asset-idiosyncratic loadings that sustains condition (ii).

Assume an aggregate market supply, $\boldsymbol{q} \in \mathbb{R}^l$, which is random and satisfies a jointly normal distribution, $\mathscr{N}(\overline{\boldsymbol{q}}, \Sigma_q)$. Random variation captures the existence of a group of "noise traders," or, as in the literature, termed "liquidity traders." Noisy trading is partly attributed to the market activities including initial public offering, stock repurchase, float adjustment by share control groups, etc. Noise traders buy an exogenous amount of shares of the assets, $\boldsymbol{q} - \overline{\boldsymbol{q}}$.

2.2 Households, managers, and information

Consider an economy where households can invest either directly or through professional delegation, typically, through funds. Thus, the economy is populated by three types of market participants: direct investors, factor investors, and fundamental investors. Factor strategies, deployed by factor funds, place bets on market factor (driven by, for example, macroeconomic announcements and overall market turbulence), and adjust exposure based on predetermined financial metrics. Fundamental investing is implemented by active equity managers, who generate excess returns from anomalies in individual security prices through informed analysis and active stock selection. Compared to fundamental investment, factor investment amounts to passive allocation, as factor funds cannot tilt their portfolios towards one particular asset.[11]

The incentives of informed trading underpin the rationale that the market is mostly, but not completely, efficient. All investors are risk averse and price takers, and, if endowed with the same information package, can be interpreted as a continuum of identical investors. I will extend the model to incorporate heterogeneous investors in Section 4. The certainty-equivalent wealth of households investing directly, or through factor or active funds, denoted by $u_j(\cdot)$ for $j \in \{h,b,a\}$ respectively, can be separated from their initial wealth W:

$$u_j(W) = -\frac{1}{\gamma} \log \left(\mathbb{E} \left[\max_{\mathbf{x}_j} \mathbb{E} \Big(e^{-\gamma \left(W + \mathbf{x}_j^\intercal (\boldsymbol{v} - \boldsymbol{p}) \right)} \Big) \Big| \mathscr{I}_j \right] \right) = W + u_j(0) \equiv W + u_j, \tag{2.2}$$

where $\boldsymbol{p} = (p_1, p_2, ..., p_l)^\intercal$ is the price vector, which will be internalized in equilibrium. I assume an identical risk-averse coefficient across all investors, $\gamma > 0$. For investor type j, $\mathscr{I}_j$ is his information set and $\boldsymbol{x}_j$ is his portfolio holding strategy, specified later in this section.

[11] Existing literature that motivates indexing in this fashion includes Cong and Xu (2016), Bond and Garcıa (2017), and Glasserman and Mamaysky (2018).

Managers

There exist imperfect signals in the market. Factor investors observe a signal of systematic risk S:

$$s = S + \varepsilon_0,$$

where S is independent of ε_0, $\varepsilon_0 \overset{d}{\sim} \mathscr{N}(0, \sigma_\varepsilon^2)$. Note that $\sigma_\varepsilon^2 = 0$ corresponds to perfect information and $\sigma_\varepsilon^2 = +\infty$ corresponds to no information. The price of the index portfolio is denoted by p_I, which, by no arbitrage condition, satisfies $p_I = \sum_{i=1}^{l} b_i p_i$, i.e., equal to the weighted sum of the prices of underlying stocks. Assume the total mass of factor funds is one.

Fundamental investors acquire asset-specific information and make investment decision at the individual stock level. Thus, besides factor loading, they also have imperfect inside information concerning idiosyncratic risks, $\boldsymbol{e} = (e_1, e_2, ..., e_l)^\intercal$, with

$$e_i = \mathscr{E}_i + \varepsilon_i, \quad i = 1, 2, ..., l,$$

where $\mathscr{E}_i$ is independent of ε_i, all ε_i are mutually independent and satisfy $\varepsilon_i \overset{d}{\sim} \mathscr{N}(0, \sigma_\varepsilon^2)$. However, as in Gârleanu and Pedersen (2019), skill acquisition is unobservable. Thus there exist "closet" funds, who opportunistically market themselves and charge fees as fundamental investors, but simply trade on overall market signals like factor investors. Assume active funds are also a continuum with mass one, among which a fraction $\mu \in [0, 1]$ is skilled, and the rest are opportunistic indexers. In an equilibrium, an endogenous amount of active managers choose to acquire skills at a cost of investigation $k > 0$.

Households

A mass λ_b of households choose factor investing, and a mass λ_a go into active funds. The rest of households, with mass λ_h, self-invest in the riskless bond and risky stocks based on public signals.

Households who favor fundamental investing are *ex-ante* unaware of which active fund manager is skilled. Thus, according to the setup, they have a probability μ of correctly picking the skilled ones, and a probability $1 - \mu$ of misallocation. But they can incur some searching cost (e.g. in information acquisition and due diligence), c, in ascertaining the manager quality. Assume there is $\xi \in [0, 1]$ fraction of households who become costly informed so that they can identify the skilled managers. The rest $1 - \xi$ households invest based on prior information. In the equilibrium, household investment in finding out manager quality is endogenous.

To sum it up, the information set of direct investors, factor investors (including closet indexers), and fundamental investors is, respectively,

$$\mathscr{I}_h = \{\boldsymbol{p}\}, \quad \mathscr{I}_b = \{\boldsymbol{p}, p_I, s\}, \quad \mathscr{I}_a = \{\boldsymbol{p}, s, \boldsymbol{e}\}. \tag{2.3}$$

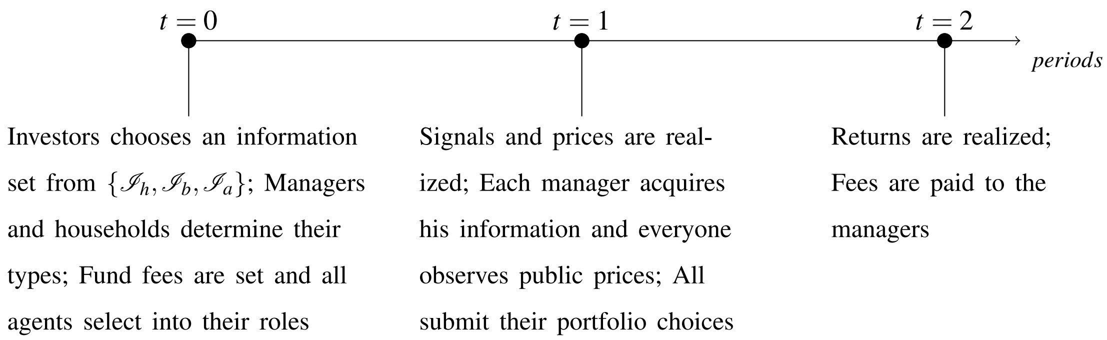
The amount of households allocating money to these three investment vehicles are, correspondingly,

$$\lambda_h, \quad \lambda_b + (1-\mu)(1-\xi)\lambda_a, \quad \big(\xi + \mu(1-\xi)\big)\lambda_a.$$

All investors fall into the three categories. Mathematically I assume

$$\lambda_h + \lambda_b + \lambda_a = 1.$$

In an equilibrium: (i) households endogenously allocate their capital between the three investment vehicles; (ii) fund managers endogenously choose to specialize in factor or fundamental information; (iii) households, if accessing the market through active funds, endogenously invest in finding out manager quality; and (iv) active managers self-select between being skilled and opportunistic. The delegation fees are endogenous and are set through Bertrand competition. The sequence of events is summarized below.



2.3 Portfolio holding strategies

Each group of investors picks his portfolio holding $\boldsymbol{x}_j$ ($j \in \{h, b, a\}$) to maximize his utility (2.2). According to the standard portfolio theory, investors pick the mean-variance efficient portfolio based on their choice of products.

Direct investors only observe the public prices and choose their portfolio correspondingly. For tractability, I assume that direct investors, in aggregation, hold the portfolio equivalent to the market portfolio. It can either because (i) they lack the expertise and skills thus, all replicate the market portfolio, or (ii) they all invest in part of the assets belonging to the index due to accessibility and aggregate to the market portfolio. Thus direct investors deploy a strategy equivalent to $\boldsymbol{x}_h = \kappa_h \boldsymbol{x}_I$, where $\kappa_h := \frac{1}{\gamma}\frac{\mathbb{E}[S|p_I]-p_I}{Var(S|p_I)}$. Factor investors profit from timing the market factor. Thus their strategy is defined by $\boldsymbol{x}_b = \kappa_b \boldsymbol{x}_I$, with the proportion $\kappa_b := \frac{1}{\gamma}\frac{\mathbb{E}[S|s,p_I]-p_I}{Var(S|s,p_I)}$. Fundamental investors actively pick their portfolio components and thus hold the following,

$$\boldsymbol{x}_a = \frac{1}{\gamma}\boldsymbol{\Sigma}_a^{-1}\left(\mathbb{E}[\boldsymbol{v}|\mathscr{I}_a]-\boldsymbol{p}\right), \tag{2.4}$$

where $\boldsymbol{\Sigma}_a$ is the variance-covariance matrix conditional on their information set, and $\mathbb{E}[\boldsymbol{v}|\mathscr{I}_a]$ is the conditional mean payoff of stocks.

Because the risky stocks are correlated both in terms of systematic loadings and idiosyncratic loadings, it becomes tricky to calculate the conditional variance-covariance matrix. Therefore I apply the Gram-Schmidt *orthogonalization* method, with details in Appendix Section B.

2.4 Equilibrium price and price efficiency

The market clearing condition is

$$\lambda_h \boldsymbol{x}_h + \left(\lambda_b + (1-\mu)(1-\xi)\lambda_a\right)\boldsymbol{x}_b + \left(\xi + \mu(1-\xi)\right)\lambda_a \boldsymbol{x}_a = \boldsymbol{q}. \tag{2.5}$$

Because investors' portfolio holding strategies are functions of prices, from the demand-supply relation (2.5) one can figure out the equilibrium asset prices, as given by Proposition 1.

Proposition 1 (Equilibrium prices) *The equilibrium prices for factor loading (p_I) and idiosyncratic loadings ($\tilde{p}_i,\ i = 1,2,...,l$), are linear in supplies and signals:*

$$\begin{aligned}
p_I &= \theta_0 + \theta_b\left[(s-\bar{s}) - \theta_q(q_I - \bar{q}_I)\right],\\
\tilde{p}_2 &= p_2 - \beta_2 p_I = \phi_a\left[(e_2 - 0) - \phi_q(q_2 - \bar{q}_2)\right],\\
\tilde{p}_3 &= p_3 - \beta_3 p_I = \psi_a\left[(e_3 - 0) - \psi_q\left((q_3 - \bar{q}_3) + \psi_\rho(q_2 - \bar{q}_2)\right)\right],\\
&\cdots
\end{aligned}$$

where $q_I \overset{d}{\sim} \mathscr{N}(\bar{q}_I, \sigma_{qI}^2)$ is the supply of S, $q_2 \overset{d}{\sim} \mathscr{N}(\bar{q}_2, \sigma_{q2}^2)$ is the supply of $\mathscr{E}_2$, and $q_i \overset{d}{\sim} \mathscr{N}(\bar{q}_i, \sigma_{qi}^2)$ for $i \geq 3$ are the rest orthogonal supplies specified in Appendix Section B. $(\theta_0, \theta_b, \theta_q)$, (ϕ_a, ϕ_q), and $(\psi_a, \psi_q, \psi_\rho)$ are parameters determined in the proof.

With *no arbitrage* property one can easily recover the price vector $\boldsymbol{p}$. In the proof, I first assume a linear form, and determine the unique value of the coefficients. Then by construction the linear price vector is the unique equilibrium price. The proof shows that, in the equilibrium, any asset component has a weight in the index proportional to its expected market capitalization. When there is a positive supply shock, the prices of both the index and individual assets fall. Thus, an additional supply will make the assets more attractive to both uninformed and informed investors.

I inherit the canonical measure of price *efficiency* from Grossman and Stiglitz (1980). Hence, the informativeness of index price is $\frac{Var(S|s)}{Var(S|p_I)}$, corresponding to the precision of information signal about the innovation to the fundamental contained in asset price. The number falls in $(0,1)$ as signal contains more information compared to price thus reduces variation. For computational convenience, I concentrate on the quantity

$$\eta_I := -\frac{1}{2}\log\left(\frac{Var(S|s)}{Var(S|p_I)}\right) \in (0,\infty), \tag{2.6}$$

termed as price *inefficiency*. Index price becomes more efficient when there are more informed trades of market factor, represented by a smaller η_I. Note that idiosyncratic information, in aggregation, does not affect the informativeness of the price of market portfolio.

Similar to index price inefficiency (2.6), I define price inefficiency for individual asset i,

$$\eta_i := -\frac{1}{2}\log\left(\frac{Var(v_i|\mathscr{I}_a)}{Var(v_i|p_i)}\right) \in (0,\infty), \quad \forall i \in \{1,2,...,l\}. \tag{2.7}$$

The informativeness of price can be directly linked to the relative performance of funds (see Appendix Section C.2). How do exogenous changes affect price efficiency will be answered after characterizing the equilibrium.

3 Asset Market Equilibrium

I have characterized the equilibrium asset price and its informativeness in the previous section. In this section, I develop the full asset market equilibrium, with the coexistence of (i) direct investors and fund users, (ii) both factor, opportunistic and skilled active managers, and (iii) households that ascertain the manager type through searching and that invest based on prior information. Without loss of generality, I focus on a model with two risky assets ($l = 2$) for tractability. Nevertheless, the rationale of the analyses prevails regardless of the number of assets. Especially, in Appendix Section E, I show that the results are applicable to a market with a large amount of assets.

The asset market equilibria will be derived following several steps. First, in Section 3.1, I derive the equilibrium fees and net-of-fee performance of funds, based on the competitive market assumption.[12] Second, I discuss the entry of skilled active managers and informed households in Section 3.2. Third, by internalizing the amount of factor investing and fundamental investing, I obtain the main results of the interior equilibrium as well as of several boundary equilibria, as

[12]This is in contrast to Gârleanu and Pedersen (2018), where the authors rely on Nash bargaining to determine fees.

presented in Section 3.3. Fourth, the estimated equilibrium conditions imply an upper bound of the fraction of capital in factor investing. The intuition is provided in Section 3.4.

3.1 Fees and performance

Funds charge fees for delegated portfolio management. These fees, in the real market, are usually a fixed fraction of the fund's assets under management, conventionally termed "management fees." Fees, in my model, will be set through Bertrand competition.

A household who delegates her wealth to factor funds, with initial wealth W, has to pay a fee α_b. Her net-of-fee utility is $W + u_b - \alpha_b$.[13] Whereas if no manager is employed, she has an outside option yielding a utility of $W + u_h$. Hence a household's net gain from factor investing is independent of her initial wealth level W. So do households choosing fundamental investing.

Similarly I can determine the fees of active managers. All active funds, skilled or opportunistic, charge an identical fee α_a, because their types are *ex-ante* unrevealed to the public. This implies that part of the managers are unfairly earning a compensation package from hidden information. Households are thus incentivized to find out about managerial capabilities.

An informed household weakly prefers fundamental investing to self investing if and only if

$$u_a - c - \alpha_a \geq u_h.$$

In an equilibrium, the informed households, after exerting cost c, are net indifferent between investing directly and with active funds, and are able to avoid closet indexers. Similar coexistence implies that households are (i) indifferent between fundamental and direct investing, (ii) indifferent between factor and direct investing, and (iii) indifferent between incurring searching cost to become informed or staying uninformed. The existence of four investment vehicles (direct, factor, closet, and skilled active investment) guarantees the following three equalities of utilities defined in (2.2),

$$u_a - c - \alpha_a = u_h, \tag{3.1}$$

$$u_b - \alpha_b = u_h, \tag{3.2}$$

$$\mu\, e^{-\gamma(u_a - \alpha_a)} + (1-\mu)\, e^{-\gamma(u_b - \alpha_a)} = e^{-\gamma(u_a - \alpha_a - c)}. \tag{3.3}$$

One can see from the above expressions that a higher gross return of fund, u_j, guarantees more

[13] If there is any concern regarding why not setting fees proportional to the realized payoff, I show here that the monetary and fractional setups are identical, yet the former greatly simplifies calculation. Suppose funds charge fractional fees $\hat{\alpha}_j$ ($j = b, a$), then the excess output to the households after fees are $(W + u_j)(1 - \hat{\alpha}_j) - (W + u_h) \approx (1 - \hat{\alpha}_j)\, u_j - u_h$. It holds approximately because the delegation fees of mutual funds are usually small. Then simply replacing $\alpha_j = \hat{\alpha}_j\, u_j$ in a framework of fractional fees, all the conclusions in my model still hold.

monetary fee payments to the j type of managers ($j = a, b$), which captures in a stylized fashion the outperforming incentive arising from the fractional fees.

To an uninformed yet risk-averse household, investing in active funds gives rise to an additional fund-selection risk. This risk is especially high when only a small fraction of managers are skilled. Thus, the expected utility from uninformed investing is not just the weighted average payoffs of picking the skilled and opportunistic managers, but with an extra premium compensating the risk of random selection. The concave combination in condition (3.3) reveals this risk premium implicitly. This is in essence identical to Berk and Green (2004), that managers on average deliver zero outperformance after fees. In this paper, however, I allow outperformance or underperformance at the individual manager level. The fact that active funds outperform the index benchmark before fees, yet not after fees, is well-documented in the empirical literature.[14]

Skilled active managers spend cost k to acquire “skill.” They subsequently benefit from attracting more capital, as they will gain money from informed individuals with certainty. Closet and factor fund managers' costs of acquiring information and formulating portfolios are normalized to zero. The fixed information costs for households, c, and for managers, k, are both exogenous. Similar to Brown and Davies (2017), I assume the fees collected by active managers are decomposed into two components: (i) a fee for the asset management service (coin the fee for factor funds), and (ii) a fee for active stock selection. Mathematically,

$$\alpha_a = \alpha_b + k. \tag{3.4}$$

Hence from the above analysis, I present the following results regarding the fee structure and performance of different types of funds, given price informativeness.

Lemma 1 (Fees and performance) *The delegation fees of factor investing and fundamental investing are, respectively,*

$$\alpha_b = \frac{\eta_I}{\gamma}, \qquad \alpha_a = \frac{\eta_I}{\gamma} + k. \tag{3.5}$$

In equilibria:

(i) Before fees, on average, active funds outperform factor investing, i.e., $\mu\, u_a + (1-\mu)\, u_b > u_b$*;*
(ii) After fees, the outperformance of active asset management disappears, typically for risk-averse households.

Moreover, the equilibrium fees, combined with condition (3.1), imply the following relationship,

$$\alpha_a = \alpha_b - c + (u_a - u_b).$$

[14] See, for instance, Jensen (1968); Sharpe (1991); Carhart (1997); French (2008); Fama and French (2010); Lewellen (2011); Busse, Goyal, and Wahal (2013).

This provides some explanation for the fact that, some active funds start to charge an option-like bonus proportional to their returns in excess of the index benchmark, mathematically, $u_a - u_b$. The above relationship does show that active fees are positively related to the excess performance of fundamental investing over factor investing.

3.2 Entry of skilled managers and informed households

The flexibility of the framework allows a mixed-strategy informed entry in an equilibrium. Taking price efficiency as given, I can figure out the equilibrium fraction of skilled managers, μ, as well as of informed households, ξ. I show that in an equilibrium, to guarantee fund flows for the overall active industry, there cannot be an excessive number of opportunistic managers.

First of all, the model shows that an endogenous amount of active managers incurs a cost k to become skilled. Condition (3.3) and previous results in Section 3.1 imply that

$$\mu = \frac{e^{\gamma k} - 1}{e^{\gamma k} - e^{-\gamma c}} \in \left[1 - e^{-\gamma k}, 1\right]. \tag{3.6}$$

One can see that μ decreases with respect to households' searching cost c. The harder it is for households to determine the managers' type, the more likely a manager opts to become opportunistic. In a hypothetical world where all households can easily find the skilled manager, all managers have to be truly skilled in order to guarantee fund inflows, i.e., $\mu \to 1$. However, in another extreme scenario, where it is impossible to determine the managers' type, not every manager free-rides. Instead, there is less, though not an insignificant amount of skilled managers, $\mu = 1 - e^{-\gamma k}$, as long as $k \neq 0$. This can be explained from the funds' perspective. First, the level of opportunism cannot be too high in order for active funds to secure inflows. Second, with most of the aggregation on the closet side, the flows left after apportion among competing funds amount to little; whereas on the skilled side it is less competitive.

Another implication is that the equilibrium fraction μ increases with the cost of fundamental information. When $k = 0$, $\mu = 0$; whereas when $k \to \infty$, $\mu \to 1$. This might seem surprising at first glance. But in this framework, the harder it is to obtain information, the more the managers are compensated by acquiring skills, implied by equation (3.4). This higher bonus facilitates a larger μ. At the same time, the high delegation cost incentivizes more households to become informed as they want to avoid selection error. Managers thus need to openly signal their superiority to secure flows from informed households. A detailed cost-benefit analysis will be provided in Section 5 of comparative statics.

Next I look into the equilibrium choice of asset managers. This helps us determine the equilibrium entry of informed households. If an active manager self-selects to become skilled, he attracts all the capital from informed households, and a fractional amount from the uninformed counterparts, i.e., $\xi\,\lambda_a + \mu\,(1-\xi)\,\lambda_a$. These amount of inflows will be shared by a mass μ of skilled funds. The rest closet funds, with mass $(1-\mu)$, receive $(1-\mu)\,(1-\xi)\,\lambda_a$ amount of uninformed capital. In the equilibrium, coexistence of the fund managers implies that (i) an active fund is indifferent between being skilled and opportunistic, and (ii) a fund is indifferent between factor investing and fundamental investing strategies. These conditions yield the following mathematical equivalence:

$$(\alpha_a - k)\,\frac{\xi + \mu\,(1-\xi)}{\mu}\,\lambda_a = \alpha_a\,\frac{(1-\mu)\,(1-\xi)}{1-\mu}\,\lambda_a, \tag{3.7}$$

$$\alpha_b\,\lambda_b = \alpha_a\,\frac{(1-\mu)\,(1-\xi)}{1-\mu}\,\lambda_a. \tag{3.8}$$

Hence, condition (3.7), after simplification, yields the equilibrium fraction of households who learn the manager type through a costly search,

$$\xi = \frac{\gamma k \mu}{\gamma k \mu + \eta_I} \in [0,1]. \tag{3.9}$$

The equilibrium fraction of informed households (i) increases with managers' information cost k, (ii) decreases with households' searching cost c, (iii) is higher when the index price is more efficient, and (iv) is higher when there are more skilled managers. However, the value domain of ξ is different from that of μ. In the extreme case where price is completely inefficient, no household is willing to become informed. Whereas if price is perfectly efficient, or the managerial qualities are easy to figure out, all households become informed.[15] Price efficiency will be jointly determined by information costs, market turbulence, capital allocation decision, and information accuracy, with details in the next section.

3.3 Equilibria

I first derive the interior equilibrium in which factor investing, fundamental investing and direct investing coexist. But it could also be the case that households strictly prefer employing fund managers or even, one specific type of managers. The discussion of these boundary equilibria and boundary conditions follows afterwards.

The three indifference conditions of households (equation (3.1) to (3.3)), the fee condition (equation (3.4)), and two indifference conditions of managers (equation (3.7) and (3.8)), are sufficient to determine six equilibrium values. Given the capital allocation strategies of households, λ_h, λ_b and λ_a, I have figured out the equilibrium fees and the amount of informed agents, $\{\alpha_a, \alpha_b, \mu, \xi\}$. Now I use the last two conditions, (3.1) and (3.8), to internalize the capital allocation. Then the

[15] In this extreme case when information of managers' type is costless, i.e., $c = 0$, all active managers have to become truly skilled ($\mu = 1$); And all households naturally become informed ($\xi = 1$). Thus condition (3.7) fails as there are no closet indexers. From $\xi = \mu = 1$ and other conditions, one can see $\lambda_b = \lambda_a$.

individual and index price efficiency are endogenously determined.

Interior equilibrium

I term the most sophisticated equilibrium, where the three investment vehicles coexist, the *interior* equilibrium. Rearranging equation (3.8) implies that

$$\frac{\lambda_b}{\lambda_a} = \frac{\gamma k + \eta_I}{\gamma k \mu + \eta_I} > 1. \tag{3.10}$$

Since $\mu \leq 1$, regardless of the equilibrium allocation, demand for factor investing is higher, i.e., $\lambda_b \geq \lambda_a$. This is consistent with the latest tendency observed in the market, that, more capital flows into the index vehicles. The interior equilibrium is characterized by Proposition 2 below.

Proposition 2 (**Interior equilibrium**) *The interior equilibrium is characterized by the following capital allocation*

$$\lambda_b = \frac{\gamma \sigma_{q_2}}{\mu} \sqrt{\frac{1}{\left(\frac{1}{\sigma_e^2} + \frac{1}{\sigma_\varepsilon^2}\right)\left(e^{2\gamma(k+c)} - 1\right)}}, \qquad \lambda_a = \frac{\gamma k \mu + \eta_I}{\gamma k + \eta_I} \lambda_b; \tag{3.11}$$

the entry of skilled managers and informed households

$$\mu = \frac{e^{\gamma k} - 1}{e^{\gamma k} - e^{-\gamma c}}, \qquad \xi = \frac{\gamma k \mu}{\gamma k \mu + \eta_I};$$

and the delegation fees

$$\alpha_b = \frac{\eta_I}{\gamma}, \qquad \alpha_a = \frac{\eta_I}{\gamma} + k;$$

where η_I is uniquely given in the proof in Appendix Section C.

One direct observation is that higher demand for factor investing is usually accompanied by more closet indexers. Under this circumstance, selection mistakes based on prior information are more likely to be made when targeting active funds. The outcome can be explained by a time-consuming process of knowledge acquisition: active mutual funds have long been believed to have the managerial skills to generate excess returns. As the outcomes from past performance and academic research become widely accepted, people in turn become more convinced of the opportunism that exists within the active industry. The positive relation is, however, not always sustained, due to the endogeneity of closet indexing. The comparative statics will be analyzed in later sections.

Furthermore, I have shown in the previous section that, as long as k is not degenerate, in the equilibrium, there exists an intermediate lower bound for the number of skilled managers, mathematically, $\mu \geq 1 - e^{-\gamma k}$. The rationale holds for all the rest boundary equilibria as well. Finally, in the proof in Appendix Section C, I show the existence and uniqueness of the solution pair (λ_b, λ_a).

Delegation dominance

When exogenous variables take extreme values, the interior equilibrium, where three types of investment vehicles coexist, fails to hold. For example, when managers' cost of producing fundamental information drops, or the trades of individual stock become active ($\sigma_{q_2}^2$ rises), professional delegations are more attractive than self-investing due to overall market volatility and fee benefits. The equilibrium thus reduces to one that is taken at an intermediate boundary, where households completely resort to delegation instead of investing themselves, i.e., $\lambda_h = 0$ and $\lambda_b + \lambda_a = 1$.

In this case, condition (3.1) and (3.2) no longer bind yet households are still indifferent between factor and fundamental investing, i.e.,

$$u_a - c - \alpha_a = u_b - \alpha_b > u_h. \tag{3.12}$$

In total, I have six new conditions including (3.4), (3.3), (3.7), (3.8), (3.12), and $\lambda_b + \lambda_a = 1$, to solve for the six unknowns $\{\alpha_b, \alpha_a, \mu, \xi, \lambda_b, \lambda_a\}$. The analytical form of the delegation-dominating equilibrium is given below. Notice that the solutions of the fraction of skilled managers, μ, and of the capital in factor investing, λ_b, remain unchanged as in the interior equilibrium.

Proposition 3 (Delegation dominate) *The equilibrium where delegation dominates is characterized by the following capital allocation*

$$\lambda_b = \frac{\gamma \sigma_{q_2}}{\mu} \sqrt{\frac{1}{\left(\frac{1}{\sigma_e^2} + \frac{1}{\sigma_\varepsilon^2}\right)\left(e^{2\gamma(k+c)} - 1\right)}}, \qquad \lambda_a = 1 - \lambda_b;$$

the entry of skilled managers and informed households

$$\mu = \frac{e^{\gamma k} - 1}{e^{\gamma k} - e^{-\gamma c}}, \qquad \xi = \frac{(2\lambda_b - 1)\mu}{(1-\lambda_b)(1-\mu)};$$

and the delegation fees

$$\alpha_b = \frac{k(1 - \lambda_b - \mu\lambda_b)}{2\lambda_b - 1}, \qquad \alpha_a = \frac{k(1-\mu)\lambda_b}{2\lambda_b - 1}.$$

When fund delegation dominates direct investing, households play a mixed strategy between factor and fundamental investing. Here, factor investing is more popular, in that we still have $\lambda_b/\lambda_a = (\alpha_b + k)/(\alpha_b + \mu k) > 1$, similar to relation (3.10). The results show that popularity in delegated asset management drives up price efficiency. This is intuitive in that there is more informed trading in the market.

Moreover, with a rise in factor investing, for instance brought by larger supply uncertainty or lower fundamental information cost, the delegation fees, especially for active funds (α_a), decrease. This is consistent with the evidence found in Cremers et al. (2016), that active funds have to charge less in order to become the effective competitors against their factor investing rivals.

More importantly, by examining the exogenous changes, I figure out the necessary condition for an equilibrium where delegation completely dominates.

Lemma 2 (**Condition for delegation dominance**) *The necessary condition of an equilibrium where delegation dominates is*

$$\lambda_b = \frac{\gamma \sigma_{q_2}}{\mu} \sqrt{\frac{1}{\left(\frac{1}{\sigma_e^2} + \frac{1}{\sigma_\varepsilon^2}\right)\left(e^{2\gamma(k+c)} - 1\right)}} \in \left(\frac{1}{2}, \frac{1}{\mu+1}\right], \tag{3.13}$$

where μ is given by (3.6).

Lemma 2 implies that when factor investing and fundamental investing coexist, there exists an upper limit for factor investing growth. This suggests that, despite the rising tendency of factor investing, it can never completely eliminate the other investment vehicles. The next section will present a clearer explanation of how the growth of factor investing is not sustainable.

Fundamental investing dominance

When fundamental information cost further reduces, or supply uncertainty or idiosyncratic risk further increases such that condition (3.13) no longer holds, the equilibrium where delegation dominates self-investing generates negative fees for factor investing, i.e., factor fund managers even provide interests to households, violating the feasibility of the equilibrium. Therefore, a new degenerate equilibrium arises, where households strictly prefer fundamental to factor investing, mathematically, $\lambda_b = 0$.

As is well known, before the emergence of indexing, people mainly counted on active equity funds or themselves to invest in risky assets. Thus an equilibrium where fundamental investing dominates factor investing is of interest with respect to the following questions: (i) What gives rise to the entry of factor investing in a world containing only fundamental investing and direct investing? (ii) Can factor investing grow immoderately to seize the entire market?

Let us first characterize the equilibrium with $\lambda_b = 0$. Because factor funds have no inflows, it becomes non-profitable for deploying factor strategies. Hence the coexistence of two types of funds, condition (3.8), fails. Coexistence of direct investment and active delegation implies that

$$u_a - c - \alpha_a = u_h > u_b - \alpha_b.$$

With the rest conditions, I character the equilibrium where fundamental investing dominates factor investing in the following proposition.

Proposition 4 (**Fundamental dominate**) *The equilibrium where fundamental investing dominates is characterized by the following capital allocation*

$$\lambda_a = \min\left[\gamma\sigma_{q_2}\sqrt{\frac{1}{\left(\frac{1}{\sigma_e^2}+\frac{1}{\sigma_\varepsilon^2}\right)\left(e^{2\gamma(k+c)}-1\right)}}, 1\right], \qquad \lambda_b = 0;$$

the entry of skilled managers and informed households

$$\mu = \frac{e^{\gamma k}-1}{e^{\gamma k}-e^{-\gamma c}}, \qquad \xi = 1;$$

and the delegation fees

$$\alpha_b = 0, \qquad \alpha_a = k.$$

From Proposition 4, one can tell that the price of the market portfolio has become perfectly efficient. Thus there is no difference between investing directly and employing factor funds. However, active funds still provide price discovery and hedging functions to households. Nevertheless, active funds cannot earn much by acquiring fundamental information when k is small, they lack the incentive to be truly skilled thus tend to free-ride. All households thus perform enough due diligence to ascertain the manager quality due to the high level of opportunism in the industry, and allocate money to the skilled managers. Upon the realization of some extreme conditions (for instance, where information production for active managers is too easy or the supply of individual assets is too volatile), households could even play a purely-active strategy in the equilibrium ($\lambda_a = 1$).

The condition to sustain the fundamental-dominating equilibrium is given in Lemma 3. The switching point of the two equilibria, may, at a first glance, be discontinuous. However, I show in the proof of the following lemma that the equilibrium solutions are always *upper-semicontinuous*. In other words, multiple equilibria coexist along the boundary line of $\lambda_b = 1/(\mu+1)$ and $\lambda_a = \mu/(\mu+1)$.

Lemma 3 (**Condition for fundamental dominance**) *The necessary condition of an equilibrium where fundamental investing dominates is*

$$\lambda_a = \gamma\sigma_{q_2}\sqrt{\frac{1}{\left(\frac{1}{\sigma_e^2}+\frac{1}{\sigma_\varepsilon^2}\right)\left(e^{2\gamma(k+c)}-1\right)}} \in \left(\frac{1}{\mu+1}, 1\right], \tag{3.14}$$

where μ is given by (3.6).

A natural concern arises that whether there exists an equilibrium where factor investing dominates the rest. In this framework, because all the agents endogenously select their roles, such equilibrium does not exist. But relaxing this requirement, the insight of this question is in essence identical to the situation modeled in Gârleanu and Pedersen (2018). With nobody employing active managers, the problem reduces to one with only two kinds of players—informed (corresponding to factor investors in my model) and uninformed (corresponding to direct investors in my model). The degenerate equilibrium is not the main focus of this paper.

3.4 Ceiling of factor investing

Following the above analysis, there indeed exists an implicit "ceiling" for factor investing: the fraction of capital in factor investing cannot exceed a fraction of $1/(1+\mu)$ of the whole market share, as implied by the estimated conditions of the delegation-dominating equilibrium and the fundamental-dominating equilibrium. The simplistic relation between λ_b and μ suggests a feasible measure, irrespective of the composition of risk appetite and information costs, to central planners concerning how much capital they shall expect to be in factor investment vehicle. Bebchuk and Hirst (2019) raise the concern that index funds control an increasingly large amount of voting shares in S&P500 companies. Regarding this as a crucial issue, the authors estimate how big can index funds grow and discuss its effect on ownership patterns in public markets from a corporate governance perspective.

In my model, the ceiling is largely determined by the opportunism level in the active industry. For active funds, being closet is a bargaining: if all households are uninformed, it is worthy; if, on the contrary, households all choose to learn manager type, then funds have to become truly skilled to retain capital. Therefore, the amount of closet funds depends both on households' searching cost and managers' information cost. So does the ceiling: the upper bound of factor investing is lower when it is harder for managers to obtain fundamental information, or when it is easier for households to detect closet indexing.

The result yields certain policy evaluations. If, regulators engage in improving market efficiency by finding out and revealing closet indexing to the public, then all active managers have to be truly skilled, i.e., $\mu = 1$. In this case factor investing could at most occupy half of the market shares. If, on the contrary, opportunistic managers exist extensively, the ceiling goes up. However, it can never reach 100%. In order for active managers to sustain inflows, there cannot be an excessive number of closet indexers, precisely, $\mu \geq 1 - e^{-\gamma k}$.

In the equilibria, households' capital allocations are determined by a bunch of exogenous variables, $\{k, c, \gamma, \sigma_{q_2}^2, \sigma_e^2, \sigma_\varepsilon^2\}$. Hence, the upper bound, $1/(\mu+1)$, is an endogenous moving surface. This can be seen from Figure A-3 in Appendix Section A, which summarizes the shift between equilibria when the exogenous variables are fixed except for information costs (k, c). Furthermore, there exist both situations where factor investing and fundamental investing move in (i) the same and (ii) the opposite direction. Hence, this model not only addresses the univariate effect of ingredients, but also explains their joint influence on the stylized facts of equilibrium capital allocation.

The model proves that multiple equilibria coexist along the boundary line of $\lambda_b = 1/(\mu+1)$ and $\lambda_a = \mu/(\mu+1)$. In fact, any combination of (λ_h, λ_b) satisfying $\lambda_b + \lambda_h = 1/(\mu+1)$ is a feasible equilibrium. Therefore, the shift between three kinds of equilibria is always smooth.

The reason behind the results is of a model interaction. The more people investing in factor funds, the more market-level information is impounded into asset prices. Upon rising to the peak level, factor investing has fully exploited the informational advantage over the systematic factor. Hence households can no longer benefit by putting money into factor funds. However, fundamental investing still has informational advantage over idiosyncratic loadings because of the great anomalies in individual security. This is expected to increase the relative gains from acquiring fundamental information, which leads to an endogenous shift from factor investing to fundamental investing. Hence, in the model, there comes a limit of how big factor investing can grow.

4 Household Heterogeneity

A fundamental insight into the asset management industry is to understand the true nature of the demand side. Heterogeneity thus matters because it generates a market where investors could have different demand schedules. To see if the allocation strategy and other equilibrium characteristics persist in a more realistic setting, in this section, I link the standard REE model to household finance. The model can help understand the effect of *household heterogeneity* on equilibrium capital allocation and the structure of the delegation industry. I show that household heterogeneity provides a possible explanation for the rise in factor investing and the compression of fund fees, a link that is testable empirically.

In the investment market, households can differ in their initial wealth and in their cognitive ability in relation to the detection of managerial skills. I hereby derive the equilibrium relaxing the identical household assumption in the baseline model. I will distinguish in my setup between *sophisticated* and *unsophisticated* households, and discuss separately their capital allocation strategies. Heterogeneity implies two main differences: (i) each individual may have specific preferences towards asset management vehicles, though in aggregation the three types of investing coexist in the market; and (ii) with heterogeneous wealth level, each individual's contribution to the funds' assets under management can differ.

I assume a continuum of households, labeled by $n \in \mathbb{N}^+$. Consider the statistical structure of different households, characterized by the set of their information cost, risk appetite and initial wealth, $\{c^n, \gamma^n, W^n\}$. Similar to the assumption in Peress (2003) that absolute risk aversion decreases with wealth, in this framework, the constant *absolute* risk aversion in exponential utility, γ^n, is determined by a following relation with *relative* risk aversion γ^R: $\gamma^n = \gamma^R / W^n$.[16] Because the difference in relative risk aversion is negligible compared to the variation in wealth, I simply assume all the households share an identical γ^R. Thus one can regard the difference in absolute risk appetite directly caused by the variation in initial wealth of households, with wealthier households less absolutely risk-averse, and vice versa. Therefore, each household can be uniquely characterized by a set $\{c^n, W^n\}$. The information cost and wealth distributions satisfy $\mathbb{E}[c^n] = c$ and $\mathbb{E}[W^n] = W$.

Following the wealth and cost distribution, the fees charged and the payoffs delivered by the funds also differ from one household to the next. Typically, wealth affects households' certainty-equivalent utility generated by the fund managers, $\{u_a^n, u_b^n\}$, because of its effect on risk appetite. Despite the above asymmetry, all households share a common price of assets, price efficiency, and funds strategies ($\{\boldsymbol{x}_a, \boldsymbol{x}_b\}$), determined by the market equilibrium. Denote by $\mathbb{E}[\frac{1}{\gamma^n}] = \frac{W}{\gamma^R} = \frac{1}{\gamma}$.

4.1 Who favors factor investing?

In this section, I defer all the detailed derivation to Appendix Section D, and provide corresponding outlines and results. Fundamental investing is a better reply to direct investing for a household if and only if

$$u_a^n - c^n - \alpha_a^n \geq u_h^n.^{17}$$

Thus the fraction of informed households, ξ, is determined endogenously by

$$\xi = \mathbb{P}(u_a^n - c^n - \alpha_a^n \geq u_h^n) = \mathbb{E}[\mathbb{1}_{\{u_a^n - c^n - \alpha_a^n \geq u_h^n\}}]. \tag{4.1}$$

Note that with a distribution of fixed cost or wealth, the market effectively has an upward-sloping supply curve, which generates smooth entry of informed households. Assume that each household has to pay an endogenous delegation fee proportional to her submitted capital. Then *ceteris paribus*, applying similar analysis as in Section 3.1 yields

$$\alpha_b^n = \frac{\eta_I}{\gamma^R} W^n, \qquad \alpha_a^n = \left(\frac{\eta_I}{\gamma^R} + k\right) W^n.$$

[16]The link between wealth and risk aversion mimics the property of the more standard CRRA utility, while maintaining the tractability of a CARA-utility model. This modeling strategy is similar to Saha (1993), Makarov and Schornick (2010), Koijen, Richmond, and Yogo (2019), and Pedersen, Fitzgibbons, and Pomorski (2019). The relation is also documented empirically by Gomez et al. (2019).

[17]The strategy within a single type of funds is identical, yet yields different certainty-equivalent wealth to individuals. Utility u_j^n ($j = h, b, a$) to household n is determined by:

$$e^{-\gamma^n u_j^n} = \mathbb{E}\left[\exp\left\{-\frac{1}{2}\left(\mathbb{E}[\boldsymbol{v}|\mathscr{I}_j] - \boldsymbol{p}\right)^\intercal \boldsymbol{\Sigma}_j^{-1}\left(\mathbb{E}[\boldsymbol{v}|\mathscr{I}_j] - \boldsymbol{p}\right)\right\}\right],$$

following definition (2.2).

Inheriting the same notation for the fraction of skilled managers, μ, the model claims that the rest of the households cannot gain from selecting an active manager based on prior information, or from investing through factor funds, mathematically,

$$\mu\, e^{-\gamma^n (u_a^n - \alpha_a^n)} + (1-\mu)\, e^{-\gamma^n (u_b^n - \alpha_a^n)} = e^{-\gamma^n (u_b^n - \alpha_b^n)} = e^{-\gamma^n u_h^n}.$$

Hence, individuals with heterogeneous wealth and cognitive ability favor different investment vehicles, with their preferences and classifications specified in the following lemma.

Lemma 4 (Who favors factor investing) *A household is classified as sophisticated if, her statistical structure $\{W^n, c^n\}$, satisfies*

$$\frac{c^n}{W^n} \leq -\frac{1}{\gamma^R} \log\Big(\frac{1 - e^{\gamma^R k}(1-\mu)}{\mu}\Big). \tag{4.2}$$

(i) The sophisticated households (ones with high wealth level and cognitive ability such that equation (4.2) *satisfies) strictly prefer fundamental investing;*

(ii) The unsophisticated households (ones with low wealth level and cognitive ability such that equation (4.2) *violates) stay uninformed and are equally likely to choose direct investing, factor investing, and allocating to active funds based on prior information.*

The outcomes of the likely composition of fundamental investing households are straightforward and intuitive, in that households with a preference for active delegation should be relatively more sophisticated with either large wealth endowment or special insights into the fund manager capabilities. In reality, for instance, institutions or rich individuals rely heavily on active mutual funds, or even on alternative investment including private equity and hedge funds. They largely exert resources to examine the managers through assessing their investment process, trading infrastructure, etc. However, small retail investors are more conservative and thus value direct investing and factor investing more. High management fees and uncertainty about managerial quality create a barrier for them to enter the active industry. As claimed by Dyck, Lins, and Pomorski (2013), it is indeed the case that sophisticated institutional investors are more likely to use active asset management compared to retail investors, as the latter group faces much higher investment costs.

Note that in the baseline model with identical households, demand for factor investing is higher than for fundamental investing, as long as they coexist. With heterogeneous households, because (i) they have different preferences towards funds, and (ii) those that choose active asset management are less wealth constrained, it could be the case that the total amount of money under management of active funds is larger than that under factor funds.

4.2 Effects on capital allocation

In contrast to the homogeneous baseline model, though factor investing, fundamental investing and direct investing coexist, different types of funds attract different kinds of households, whose wealth affects the ultimate payment to the managers. Skilled managers receive all of the inflows from the sophisticated households, and part of the inflows from the unsophisticated households; whereas factor and closet funds, as given in Lemma 4, only attract the uninformed and unsophisticated households. Hence the equilibrium conditions of fund managers become

$$\mathbb{E}[\alpha_a^n W^n|\text{uninformed}]\,\frac{(1-\mu)(1-\xi)}{1-\mu}\,\lambda_a = \mathbb{E}[\alpha_b^n W^n|\text{uninformed}]\,\lambda_b = \left(\mathbb{E}[\alpha_a^n - kW^n|\text{informed}]\,\frac{\xi}{\mu} + \mathbb{E}[\alpha_a^n - kW^n|\text{uninformed}]\,\frac{\mu(1-\xi)}{\mu}\right)\lambda_a.$$

With these indifference conditions, one can solve the equilibrium system. λ_a and μ are both functions of price inefficiency η_I. ξ can be uniquely mapped to μ, thus is also a function of η_I. Hence I determine the equilibrium price efficiency first, then all the rest terms follow. The detailed derivation is deferred to Appendix Section D.

I now examine how household heterogeneity affects the equilibrium capital allocation, fees, and opportunism level in the economy. As we are faced with a joint distribution of two variables, wealth level and cognitive ability (information cost), one needs to specify the distribution to model the effects.

Wealth inequality

First, fix the cognitive ability dimension and consider wealth inequality. Assume a fixed information cost for every household, $c^n \equiv c$. To model the dispersion of wealth, I restrict the distribution of wealth, W^n, on a fixed support, $[0, 2W]$. Now I fix the expectation of the distribution to be W, and vary the variance and shape of distribution to capture the dispersion. Without loss of generality, I assume the wealth variable W^n satisfies the Beta distribution on the finite positive support $[0, 2W]$, with density function as follows,

$$\text{Beta}(a,b) = \frac{(\frac{x}{2W})^{a-1}(1-\frac{x}{2W})^{b-1}}{2W\,B(a,b)}, \qquad \text{where } B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a,b)},$$

where B stands for the Beta function and Γ stands for the Gamma function. The mean of this distribution is $2W\,a/(a+b)$. To focus on the dispersion, I fix the mean at W by assuming $a = b$. When $a = b < 1$, the distribution is bimodal (or "U-shaped") thus is termed highly dispersed; whereas when $a = b > 1$, the distribution is unimodal, with a higher a implying a more concentrated distribution. In the intermediate scenario where $a = b = 1$, the distribution becomes uniform. In a nutshell, the smaller a is, the more unequal the wealth distribution is. Furthermore, Gini coefficient is a well-known measure for wealth inequality, which, given any wealth distribution, can be calculated theoretically.[18] Thus more intuitively, I map the wealth distribution to Gini index, and use it as the main measure of statistical wealth dispersion. See Figure D-4 in Appendix Section D.3 for an intuitive summary.

One can tell the following equilibrium features brought by wealth inequality.

Lemma 5 (**Wealth inequality**) *As the wealth level among households becomes more dispersed:*

(i) More managers become skilled;
(ii) Fewer households become informed towards manager quality;
(iii) The percentage delegation fees decrease for both factor and active funds;
(iv) The demand for factor investing increases;
(v) Prices are more informative.

Somewhat surprisingly, when the population in a society has a larger wealth gap, the demand for factor investing is higher. By unraveling the rationale behind this, one begins to detect some logic. First, wealthier households benefit from searching for fundamental investors, since their searching cost is low relative to their capital. Second, heterogeneity concentrates wealth in the hands of less marginally risk-averse wealthy households, whose wealth effect brings the individual prices closer to their fundamental values, shrinking the outperformance of fundamental investing. At the same time, the large possible inflow from sophisticated households makes it more attractive for managers to become skilled. Thus many active managers rush to acquire relevant skills, thereby increasing the inside competition of fundamental investing and curtailing the return of active strategy even further. The equilibrium outcome of decreasing returns to scale provides justification for such assumption in Berk and Green (2004) and Pástor and Stambaugh (2012).

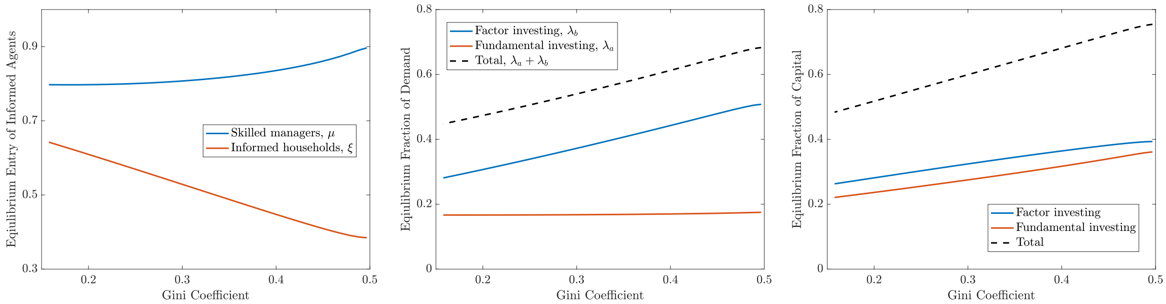
Therefore, we observe simultaneously that: (i) active funds lower their fees to remain competitive; and (ii) people become less interested towards fundamental investing. The aggregation effect is, at first glance, ambiguous. But the model shows that the latter effect dominates: more households thus, instead, choose to stay uninformed and embrace the cost-efficient factor investment vehicle, represented by a smaller ξ. Furthermore, the reduced fees earned by active funds force factor funds to lower their fees as well, leaving less wealthy households less likely to directly invest and more likely to opt for delegation.

Active funds, on the one hand, are directly harmed by the decrease of informed households; on the

[18] Gini coefficient is a value taken within the region $[0,1]$. A coefficient of 1 is interpreted as maximal inequality and a coefficient of 0 means households have completely identical wealth. Gini index has long been used to measure wealth inequality, see, for instance, Yitzhaki (1979) and Lambert and Aronson (1993).

other hand, they can attract more of the random flows from the uninformed households because of the fee advantage. Hence, in aggregation, the demand for fundamental investing remains roughly unaffected, suffering only a tiny decrease. The rise in demand for factor investing is mainly compensated by direct investing.

On the contrary, when most households are middle class, factor investing diminishes, and converges to the limit of λ_b in the homogeneous-wealth equilibrium (if $W^n \equiv W$ for any n), given by equation (3.11). For clarity, a numerical example is given in Figure 1. With a reasonable choice of parameters, in the example the delegation fees decrease by about 0.5% for both funds as the wealth distribution turns from equal to unequal.[19]



A. Informed agents. B. Demand. C. Capital.

Figure 1: *Wealth inequality on capital allocation.* I vary the parameter of Beta distribution, a, in $(0,6]$ to get the corresponding Gini coefficient (x-axis), holding the expected wealth $W = 8$ and the support of wealth distribution $[0,16]$. For the other parameters, take $\gamma^R = 3$, $k = 0.05$, $c = 0.1$, $\sigma_{q_1}^2 = \sigma_{q_2}^2 = \sigma_e^2 = \sigma_s^2 = 2\,\sigma_\varepsilon^2 = 0.5$.

Panel B of Figure 1 confirms the previous analysis of increasing demand for factor investing. Nevertheless, due to wealth effects, the money put into funds by rich and poor households carry different levels of power. Thus the total capital, in monetary terms, pouring into funds differ from one household to the next. Skilled active funds have a wealthier investor base. The model shows that the outcome of the rise in factor investing is robust in terms of capital, and that heterogeneity amplifies the fraction of capital in both funds (Panel C). In other words, wealth inequality among households increases the assignment of professional delegation while curtailing uninformed direct investment.

In the past decades, tendencies of increased factor investing, decreased delegation fees and increased wealth inequality have all been observed. Typically, for instance, the U.S. investment market has been experiencing these changes simultaneously, as can be seen from Figure A-1 and A-2. This model to some extent ties the three observations together and explores the possible causal link between these trends over time.

[19] The numerical level of relative risk aversion is consistent with that in Gârleanu and Pedersen (2019).

Heterogeneous cognitive ability

Then I look into the other dimension—heterogeneous cognitive ability among households, following the same rationale. By high cognitive ability, I mean the household can incur a low cost to become aware of the manager quality. This time I fix the wealth level, $W^n \equiv W$, and model the cost dispersion on a fixed, finite, and positive support $[0, 2c]$. Without loss of generality, I assume $c^n \overset{d}{\sim} \text{Beta}(a, b)$ with $a = b$. Thus the expected cost of learning the managers' type is fixed at c. A small a stands for asymmetric insight of manager skills. In real world, it is highly possible that institutions and professional investors have more insight towards which active manager is better.

One can tell the following equilibrium features with households heterogeneity in cognitive abilities, as given in Lemma 6.

Lemma 6 (**Cognitive ability dispersion**) *As the cognitive ability among households becomes more dispersed:*

(i) Fewer households become informed;
(ii) Fewer managers become skilled;
(iii) Demand/Capital switches a little bit from fundamental investing to factor investing.

A numerical example is given in Appendix Section D.3. In Grinblatt, Keloharju, and Linnainmaa (2011), the authors find that high-IQ households are more likely to delegate to funds and hold a large number of stocks. In this model, I enrich the analysis by looking into the discrepancy in cognitive ability and its effect on delegation in the intensive margin. The model shows that a larger inequality in cognitive ability gives rise to factor investing. This tendency can be explained by a higher level of opportunism among fundamental investors, which prevents households from being informed and delegating money to such investors, as represented by a smaller ξ. However, the implication is not as clear here as with wealth inequality, and the effect is not as large. Furthermore, an asymmetric cognitive ability across households amplifies the average outperformance of fundamental investors in equilibrium. This is because a decreasing amount of skilled active managers lowers the inside competition in fundamental investment.

By examining a series of reasonable exogenous variable choices, I conclude that, in this framework, the wealth effect dominates the cognitive ability effect in terms of magnitude. However, there exist other possibilities in relation to modeling cognitive ability. For instance, it is highly likely that households with different cognitive capability have different certainty-equivalent wealth generated by funds. The possibility that cognitive ability affects utility is not incorporated in my model. Had this been included, it is possible that other effects, maybe more magnificent, on the overall asset market equilibria brought by cognitive capability could have emerged.

Unequal society

I analyze independently the two dimensions of heterogeneity in the previous sections. Now I examine the case where households' wealth and cognitive capability are correlated. The link between the two, in the existing literature thus far, is a bit mixed. Rindermann and Thompson (2011) find some extent of positive correlation, whereas Zagorsky (2007) find a quadratic relationship. In this framework, it is highly likely that one with higher wealth endowment also has higher cognitive ability (or, lower information cost). For instance, if one thinks of "households" in a broader sense to include institutions, pension funds, endowments and sovereign wealth institutes, these asset owners aggregate large amount of wealth, and are highly likely to have more insight into the fund quality. I term such a society with positive correlation between wealth and cognitive ability an *unequal* society. Equivalently, it can be captured by a negative correlation between wealth and information cost level.

For this purpose I take one of the Archimedean copula—Ali-Mikhail-Haq copula (AMH for short, see Ali, Mikhail, and Haq (1978))—as the joint distribution of (W^n, c^n), which could capture both positive and negative correlations. Restricting the marginal distribution of wealth as uniform on $[0, 2W]$, and the marginal distribution of information cost as uniform on $[0, 2c]$, the AMH copula implies the following cumulative distribution function:

$$\mathrm{AMH}(\theta) = \mathbb{P}(W^n \leq x, c^n \leq y) = \frac{xy}{4Wc - \theta\,(2W - x)\,(2c - y)}, \qquad \text{where } \theta \in [-1, 1].$$

The parameter θ captures the correlation between wealth and information cost. A negative θ implies a negative correlation, intuitively, wealthier households have easier access to the managerial quality information; A positive θ implies a positive correlation; And $\theta = 0$ means that wealth and cognitive ability are independently uniformly distributed. Note that, however, $\theta = 1$ (-1) is not translated to perfectly positive (negative) correlation, but a one to one mapping to some intermediate correlation level.

The asset market equilibrium is influenced by the correlation in the following way.

Lemma 7 (**Unequal society**) *As the society becomes more unequal:*

(i) More managers become skilled;
(ii) Fewer households become informed towards manager quality;
(iii) The percentage delegation fees decrease for both factor and active funds;
(iv) The demand for factor investing increases;
(v) Prices are more informative.

The intuition and outcomes are very much similar to the case with one-dimensional dispersion in wealth. A numerical example summarizes the findings, as provided by Figure D-6 in Appendix Section D.3.

4.3 Dynamics of opportunism

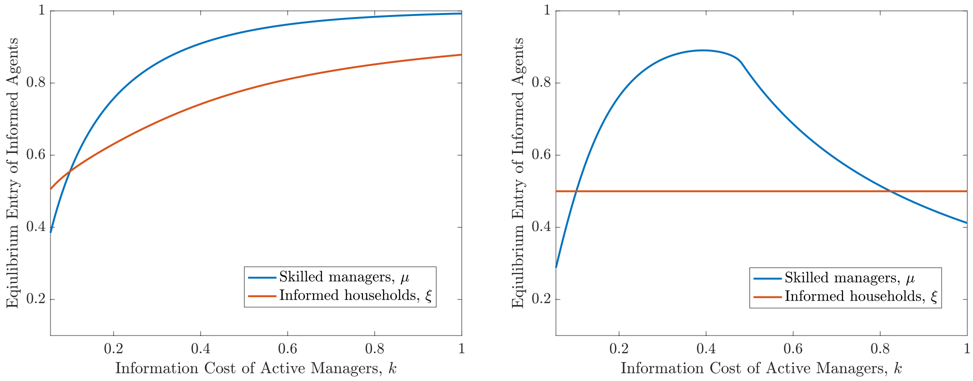
Heterogeneity, or an unequal society, can help explain other pertinent features of the asset market equilibria, including the dynamics of opportunistic managers.

Consider a discrete example below, with only two different types of households. The first group of households are relatively rich (with wealth $\overline{W}$) and all become informed due to their easy accessibility to inside information of manager type (for instance, they share a searching cost normalized to $\underline{c} = 0$). The second group owns limited wealth ($\underline{W}$) yet an extremely high searching cost ($\overline{c}$), thus are never able to learn the type of active managers. The market consists of mass ξ of the first group and mass $1 - \xi$ of the second group. Lining up with the previous terminologies, the first group coins the sophisticated households and the second group unsophisticated. The Bernoulli distribution exogenously determines the entry of informed households, ξ. Any exogenous change in the investment market, including frictions or supplies, cannot affect the amount of informed households. A skilled manager cannot attract more households through openly signaling his type, either. Typically, in the following example, I show that a variation in the information cost of fund managers, k, though not affecting household partition, has interesting implications on the dynamics of opportunism in the active industry.

Example 1 *As shown in Lemma 4, the group of sophisticated households always prefer fundamental investing, as they can target the "right" managers. While the unsophisticated group who lacks both money and insight, randomizes between directly investing, factor investing or picking active funds based on prior information. If they choose active delegation, there is a certain probability $1 - \mu$ that their money goes to an opportunistic manager. Holding the average wealth and information cost consistent with the homogeneous model, I compare the change of informed agents with respect to managers' information cost k, as shown in Figure 2.*

Interestingly, in such an extremely unequal society, the most efficient level of active industry—where largest amount of managers are truly skilled—becomes intermediate as a function of k. I term the market with heterogeneous households a "talent market," where a fixed fraction of sophisticated households knows clearly the managers' type, yet the rest are prevented from accessing such information due to their high searching cost. The amount of talented households, ξ, is exogenous. The homogeneous equilibrium, however, happens in an "effort market," where every household is identical and can exert some effort to search for managers, in which case, endogenously determines ξ.

In the effort market, as the information cost of managers, k, increases, the managers also charge



A. Homogeneous. B. Heterogeneous.

Figure 2: *Comparison between homogeneous and heterogeneous market.* As I vary the choice of k in $(0,1]$, this figure shows the change of equilibrium amount of skilled managers and informed households, with Panel A for homogeneous equilibrium and Panel B for heterogeneous equilibrium. I keep mean wealth and cognitive capability the same in two models: $\mathbb{E}[c^n] = \frac{1}{4}$ and $\mathbb{E}[W^n] = 2$. In Panel B I assume $\overline{c} = 0.5$, $\underline{c} = 0$, $\overline{W} = 3$, $\underline{W} = 1$. For the other parameters, take $\xi = 0.5$, $\gamma^R = 4$, $\sigma_{q_1}^2 = \sigma_{q_2}^2 = \sigma_e^2 = \sigma_s^2 = 2\,\sigma_\varepsilon^2 = 0.2$.

more to compensate the cost. Thus households who choose active delegation need to be more confident in picking the right managers. Hence, the entry of informed households is monotonically increasing with respect to k. The large amount of informed households also drives up the amount of skilled managers, because they want to openly signal their type in order to attract money from the informed households. The results are illustrated in Panel A of Figure 2.

In contrast, in the talent market, the monotonicity no longer exists, as shown in Panel B. When information cost k is relatively small for the active managers, they do benefit from acquiring skills and charging a premium accordingly. Nonetheless, at some point they start to give up the inflows from the talented households, in the model, μ shrinks. This is because the new inflows, after being shared by a large fraction of competing skilled funds, cannot offset the high cost of acquiring skills. They also need to generate large enough gross returns to justify the high fees. Because of the exogeneity of ξ, they can never incentivize the unsophisticated households to become informed. This is intrinsically why acquiring skills ceases to be attractive for managers in the heterogeneous world. As a consequence, μ is hump-shaped.

The outcome above, that the entry of skilled managers differs in the *effort* and *talent* market, is intuitive. In the effort world, μ is endogenous but uniquely determined by the information frictions:

$$\mu = \frac{e^{\gamma k} - 1}{e^{\gamma k} - e^{-\gamma c}}.$$

The variation in k generates a monotonic effect on μ. A higher fundamental information production cost allows active managers to charge higher premium and to benefit from information acquisition. Hence being a fundamental investor becomes more attractive.

In contrast, in the talent world, μ is a function of both information frictions and price efficiency. For instance, in the above extreme example with Bernoulli-distributed statistical structure of households,

$$\mu = \frac{\eta_I}{k\gamma^R}\frac{\overline{W}\,\xi}{\underline{W}\,(1-\xi)},$$

where k influences μ through (i) influencing price efficiency, and (ii) influencing μ directly. When k increases from a low level, there is high demand for factor investing, forcing η_I to be low. The highly informative index price provides space for fundamental investing to profit. Thus managers are incentivized to become skilled. However, a too large k prevents households from using funds and gives birth to opportunistic managers who try to attract part of the capital without incurring the high cost. Thus there exists complex involution between k and μ, causing the monotonicity to vanish.

In fact, in a continuous talent market, where wealth and information cost satisfy a continuous joint distribution, the rationale of analyzing μ is similar as in the discrete example. The entry of skilled managers will be determined through finding the fixed point of the following equation:

$$\frac{\eta_I}{\gamma^R\,\mu}\,\mathbb{E}[W^n|\text{informed}]\,\xi = k\,\mathbb{E}[W^n|\text{uninformed}]\,(1-\xi),$$

where ξ is no longer exogenous yet is uniquely determined by μ (details in Appendix Section D). Thus given η_I, the above expression uniquely determines μ, in a similar fashion as in the discrete example. In a nutshell, household heterogeneity makes the structure of the active industry more intertwined with the market conditions, thereby alters the dynamics of equilibrium capital allocation and delegation fees.

4.4 "Alpha"

Household heterogeneity has certain implications on the outperformance of fundamental investing over factor investing. One canonical way to estimate a skilled active fund's excess return over market is through the Capital Asset Pricing Model (CAPM) and look into the "alpha" it generates, i.e., the intercept of the Security Market Line (SML). In this model, the only group of participants who can capture the abnormal return is the group of skilled active managers, who yield inside information towards idiosyncratic loadings and profit through rigorous stock selection. They are also known for charging higher fees due to the alpha they generate.

Based on the model setup, the risk-appetite-adjusted outperformance of skilled active funds over factor funds, $\gamma^n\,(u_a^n - u_b^n)$, becomes a tractable measure that captures the excess performance in a stylized fashion.[20] From the above analysis, the abnormal return of fundamental over factor

[20] In Appendix Section E, I provide the rationale to properly analyze "alpha" in CAPM.

investing can be linked to the equilibrium amount of skilled managers as

$$\gamma^n \left(u_a^n - u_b^n\right) = \gamma^R k - \log\left(\frac{1 - e^{\gamma^R k}(1-\mu)}{\mu}\right).$$

Because the relative risk aversion and managers' information cost are exogenous, the outperformance level varies in the opposite direction of μ, another endogenous variable. As μ increases, the more efficient prices and the higher competition among skilled managers lead to an endogenous decreasing returns to scale of fundamental investors. Therefore, the highest possible excess return generated by fundamental investing is when there is least competition within the industry.

5 Drivers and Effects of Rising Factor Investing

5.1 Why factor investing rises?

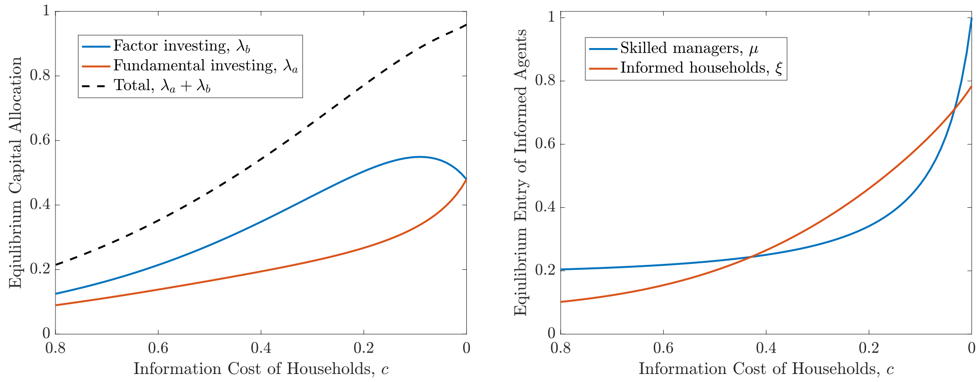
The analysis of asset market equilibrium and household heterogeneity provides one possible explanation for the rise in factor investing. In this section, I summarize some other possible drivers of factor investing through analyzing the comparative statics of the equilibria. These can help understand the empirical observations as well. Given the analytical equilibrium solution of demand for factor investing in Proposition 2, I present a preview of the contributes in the following proposition, and discuss the intuition in the following sections.

Proposition 5 (Drivers of factor investing) *In the interior equilibrium, demand for factor investing:*

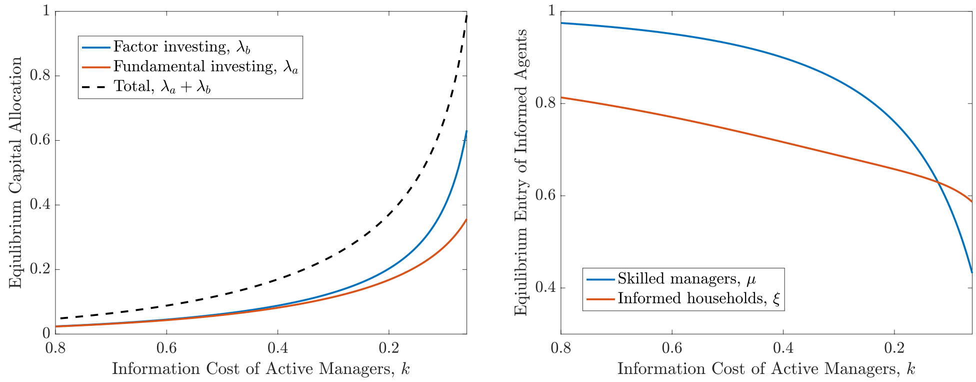
(i) increases then decreases with household searching cost c;
(ii) decreases with managers' cost of fundamental information k;
(iii) increases then decreases with investor risk aversion γ;
(iv) increases with idiosyncratic risk σ_e^2 and supply uncertainty σ_{q2}^2;
(v) increases with household inequality.

Information costs

In general, a drop in information costs, either c or k, increases the demand for fund delegation, which is compensated by a decrease in direct investment. However, despite this general tendency, the detail in and rationale behind how the two types of information cost influence capital allocation are very different. Specifically, the two types have opposite effects on the level of opportunism in the economy. The comparative statics can be found in Figure 3.



A. Households' information cost, c.



B. Managers' information cost, k.

Figure 3: *Comparative statics w.r.t. households' and managers' information cost.* As I vary $c \in [0, 0.8]$ while fixing $k = 0.07$ (Panel A), or vary $k \in (0, 0.8]$ while fixing $c = 0.1$ (Panel B), this figure shows the equilibrium capital allocation, and amount of skilled managers and informed households. For the other parameters, take $\gamma = 3$, $\sigma_{q_I}^2 = \sigma_{q_2}^2 = \sigma_e^2 = \sigma_s^2 = 2\sigma_\varepsilon^2 = 0.2$.

First, let us take a look at the searching cost of finding out the managers' type. The demand for factor investing is hump-shaped with respect to the change of c. A general decrease in searching cost c has the following two effects: (i) it makes opportunistic strategy harder, curtailing the amount of closet indexers in the economy; and (ii) it shrinks the outperformance of fundamental investing compared to factor investing (from equation (3.1) and (3.2) one can conclude that $u_a - u_b = k + c$). In turn, fund managers have to charge less, leading to compression of both fund fees. Therefore, households are more likely to delegate their money to funds in general, especially the cost-efficient factor investing vehicle. Ultimately, more informed trading improves price efficiency, both for the index and individual assets. However, when searching cost c further decreases, this regime may backfire. A sufficiently small cost c leads to a sharp increase in the amount of skilled managers, leading households to shift to fundamental investing as a "substitute" due to the high efficiency in the active industry. In the extreme case that households can ascertain managerial quality without any cost ($c \to 0$), all active managers have to acquire skills in order to attract inflows ($\mu \to 1$). Thus the efficiency of the active industry drives its inflows to the maximum level, which is equal to the flows of factor funds, i.e., $\lambda_b = \lambda_a$.

Second, for comparison, I analyze the effect of managers' cost of obtaining fundamental information, k. With a large k, fundamental investors struggle to produce inside information. Thus, they charge more and these high expenses block most households from engaging professional delegation. Therefore, in the feasible region of the interior equilibrium, a natural outcome is that the decrease in managers' information cost spurs more fund delegation. When k is relatively small, active funds do not gain much for becoming skilled. This is because the highly efficient asset prices shrink the profitable region of fundamental investors, making them reluctant to become skilled. Due to the high level of opportunism and the tiny gap between the fees for fundamental investing compared to factor investing, there is a not insignificant number of households willing to find and vet the skilled managers and delegate money to them. Note that I do not emphasize any cases where k is too small, in order to avoid the equilibrium being reduced to the boundary ones. The boundary equilibria have been discussed in Section 3.3.

In today's reality, with the help of the internet, information regarding asset management services has become more transparent. On the one hand, it is easier for households to acquire information concerning fund managers' investment principles and historical performance, which helps households to target competent managers more precisely. On the other hand, managers share more expertise and accessibility to discover the fundamental values of companies. The emergence of electronic trading and data mining has greatly reduced the friction for managers to trade and acquire information. Therefore, it can be anticipated that demand for factor investing is growing.

Lastly, the information frictions directly affect the level of opportunism in the active industry (equation (3.6)). Straightforwardly, demand for factor investing is higher with a higher level of opportunism in the active industry. This can be explained from the following two perspectives. First, due to the popularity of active management in the past half century, the funds engage in fierce competition with each other, and share high levels of similarity in their strategies.[21] These lead to the decrease in the number of managers who can stand out and still deliver stellar returns compared to their peers. Second, active funds have long been believed to adopt skills enabling them to generate excess returns. But as more and more performance analysis and academic research emerge, households are now more aware of which funds are truly competent, and are also more convinced of the existence of free-riders in the industry. This trend could possibly lead to a smaller μ and more people opting for factor investing as a safer substitute.

Risk appetite

I examine how changes of risk appetite affect the asset market equilibrium. The model predicts that both factor investing and fundamental investing reach peak level at an intermediate risk aversion level, contradicting the argument in Glasserman and Mamaysky (2018).

[21] See, e.g., Chevalier and Ellison (1999) and Hong, Kubik, and Stein (2005).

As can be seen from Figure 4, the equilibrium capital allocations are of a concave and humped pattern, as the investors become more risk-averse. Funds in essence hold inside information concerning the volatility of either systematic or idiosyncratic loadings. When $\gamma \to 0$, the investors are inclined to be risk-neutral, whereby, expected returns become the key focus, while volatility and signals play only a very limited role. At the same time, asset prices become highly informative because the prices are more sensitive to supply shocks. Hence, the capital allocated to funds is diminishing and households instead prefer direct investment. When γ becomes too large, households are highly risk-averse, thus they gradually shift from risky assets to riskless bond, causing a decrease in fund delegation as well. Thus there exists an intermediate maximal of demand for delegation, where both conditions are satisfied: (i) households are willing to hold risky assets, and (ii) they care about risk, which is characterized by volatility. Note that the maximal demands for factor and fundamental investing are not reached at the same risk aversion level.

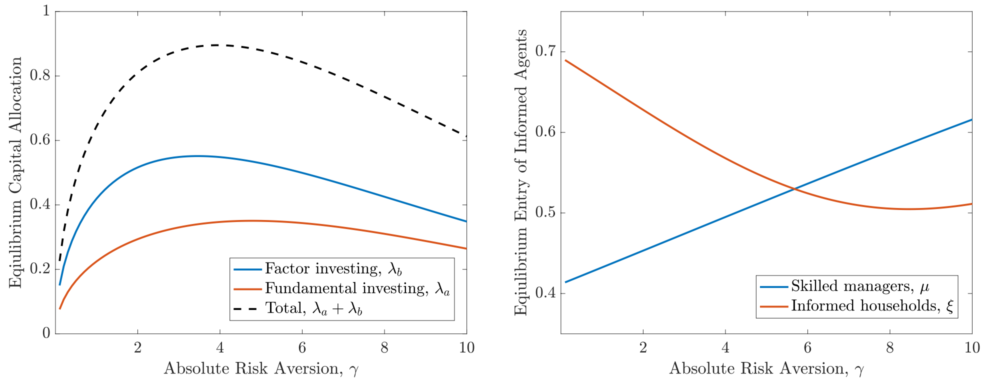


Figure 4: *Comparative statics w.r.t. investors' risk appetite.* As I vary the choice of γ in $(0, 10]$, this figure shows the equilibrium capital allocation, and amount of skilled managers and informed households. For the other parameters, take $k = 0.07,\ c = 0.1,\ \sigma_{q1}^2 = \sigma_{q2}^2 = \sigma_e^2 = \sigma_s^2 = 2\sigma_\varepsilon^2 = 0.2$.

I also find that the fraction of informed households, ξ, is of a convex shape with respect to risk appetite. Information is more valuable when there are fewer skilled managers, typically, when risk-aversion is low. At a relatively high level of risk-aversion, households hold fewer risky assets. This is when the cost of investing in wrong risky assets becomes extremely high. Thus more households incur some searching cost to detect closet indexing. Moreover, though not shown in the figure, high risk aversion harms price efficiency. The decreasing price efficiency is consistent with the prediction in the Kyle (1985) model, where higher levels of risk aversion lead those who are informed to trade less, engendering a less informative price.

As households gradually become more aware of the assets, risks, and investment tools, they develop a more comprehensive view of wealth management. Instead of putting money into banks, they become more willing to hold risky assets. At the same time, they care about both risk and return. This development, in my model, can be translated into an intermediate level of absolute risk aversion γ.

Supply uncertainty

An unstable environment of individual assets, including return volatility, supply uncertainty and information accessibility, will affect people's delegation choices. If the asset-specific risk becomes higher (σ_e^2 increases), or the information about idiosyncratic loadings becomes less precise (σ_ε^2 increases), or the supply becomes highly uncertain (σ_{q2}^2 increases), factor investing becomes more attractive, because systematic risk is relatively easy to control. For risk-averse households, factor investing delivers a safer and more stable payoff compared to fundamental investing. At the same time, fundamental investors can still attract greater demand because they partly profit from the idiosyncratic risks.

I take the supply uncertainty as an example. More noise trading leads to higher supply uncertainty of assets. As I vary the supply uncertainty of individual asset σ_{q2}^2, the equilibrium capital allocation is shown in Figure 5. The figure does not cover the cases where the supply is too volatile. This is when demand for factor investing becomes non-monotonic, as it reaches the ceiling and has fully exploited the informational advantage over market factor.

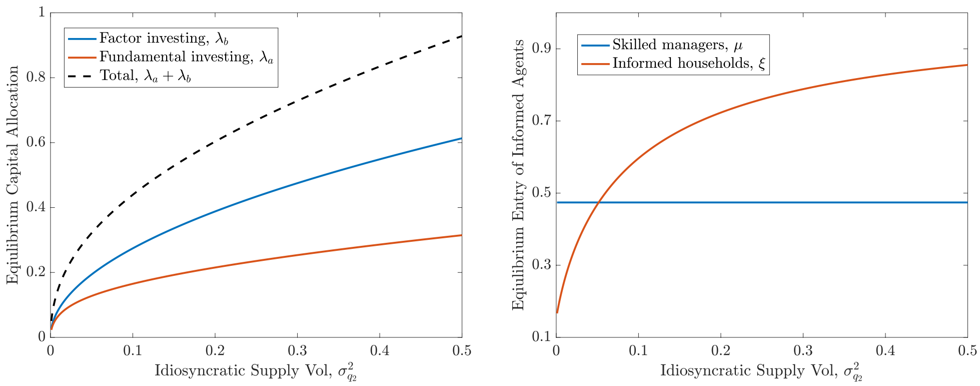


Figure 5: *Comparative statics w.r.t. supply uncertainty.* As I vary the choice of σ_{q2}^2 in $(0, 0.5]$, this figure shows the equilibrium capital allocation, and amount of skilled managers and informed households. For the other parameters, take $k = 0.07,\ c = 0.1,\ \gamma = 3,\ \sigma_{q1}^2 = \sigma_e^2 = \sigma_s^2 = 2\sigma_\varepsilon^2 = 0.1$.

A high supply uncertainty drives up the capital in the overall delegation industry, including both factor and fundamental investing, and curtails the delegation fees. Skilled managers do benefit from signaling their superiority because of an upward sloping curve of informed households. However, the decreasing fees deteriorate their gains directly. In aggregation, the amount of skilled managers remains unchanged. Interestingly, the impact of idiosyncratic supply uncertainty on capital allocation cancels out its effect on the price informativeness of idiosyncratic loading itself. Hence, when there is a supply shock, the size of the shock, σ_{q2}^2, affects mainly the price efficiency of systematic loading. This is because part of the effect of idiosyncratic supply uncertainty is internalized by the market itself through altering the allocation preference of households.

5.2 Effects on return volatility and price efficiency

I now investigate the change of price terms when the capital in factor investing is larger. First I give some theoretical hints as to how to analyze the return volatility and price efficiency.

An increase in factor investing can be compensated by a decrease in either (i) fundamental investing or (ii) direct investing. Thus I define two ratios. The first ratio is the amount of factor investing to fundamental investing, i.e.,

$$r_{f-f} = \frac{\lambda_b}{\lambda_a}. \tag{5.1}$$

If the rise of factor investing does not affect direct investing much, yet curtails fundamental investing, then r_{f-f} would rise. The second ratio is the wealth between non-fund users and fund users, i.e.,

$$r_{n-f} = \frac{\lambda_h}{\lambda_b + \lambda_a}. \tag{5.2}$$

If an increase in factor investing curtails direct investing, r_{n-f} would drop. Therefore, to examine the influence of increased factor investing, I examine how the price terms change with respect to the two ratios defined by (5.1) and (5.2).

Provided with the equilibrium price specified in Proposition 1, I look into its changes in value, stability, and co-movement brought by rising factor investing, as shown in Lemma 8.

Lemma 8 (**Factor investing and price**)

(i) If increased factor investing is compensated by less fundamental investing (r_{f-f} rises):
- *(a) The price of asset-idiosyncratic component becomes more volatile;*
- *(b) The covariance between the prices of asset-idiosyncratic components increases;*
- *(c) The return of asset-idiosyncratic component becomes more volatile.*

(ii) If increased factor investing is compensated by less direct investing (r_{n-f} drops):
- *(a) The expected prices increase;*
- *(b) The index price becomes more volatile, if the inside information is precise enough;*
- *(c) The covariance between individual prices increases, if trading of the index portfolio is liquid enough.*

The proof of Lemma 8 shows that the increasing instability, given by result (ia), is mainly attributed to the idiosyncratic component of an asset. The price could also be more volatile for the index in certain scenarios, as stated by (iib). Second, factor funds all feature the same bunch of portfolio, in which sense the investment market becomes highly concentrated through the popularity of factor investing. The model shows that a more concentrated industry is more vulnerable to idiosyncratic supply shock. Third, the rise in factor investing increases the price co-movement of assets in the index, given in (ib). In fact, one can tell from the proof that the influence of r_{f-f} on variance and covariance of assets is not linear but quadratic. The analysis lends support to the recent regulation reports, Sushko and Turner (2018) and Anadu et al. (2018), highlighting that fundamental-to-factor shift may affect financial stability. The co-movement result also lines up with the model extension outcome in Basak and Pavlova (2013).

Finally, the idiosyncratic risk premium becomes more volatile as λ_b increases. This is where active funds benefit from, in that the excess utility generated by fundamental over factor investing managers is positively correlated with the risk premium uncertainty.[22] Thus a byproduct of this conclusion is the increase in the outperformance of skilled active over factor funds. If the rising factor investing is accompanied by a decrease in fundamental investing, the remaining active managers benefit from the less competitive environment thus are more likely to generate superior performance. The intuition of this result is provided by a FCA occasional paper (James et al., 2019).

Increased factor investing has impact on the amounts of informed trading in different portfolios, which, for sure, will affect the information embedded into asset prices.

Lemma 9 (**Factor investing and price efficiency**)

(i) If increased factor investing is compensated by less fundamental investing (r_{f-f} rises), the price efficiency of individual assets decreases (represented by an increasing η_i, $i = 1, \dots, l$);

(ii) If increased factor investing is compensated by less direct investing (r_{n-f} drops), the price efficiency of market index increases (represented by a decreasing η_I).

The influence of r_{f-f} on the information efficiency is neither linear nor quadratic, yet of a more complex relation as stated in the proof. Internalizing the amount of factor investing, I summarize a few possible changes that can harm price efficiency in Figure 6. I recover from definition (2.6) and (2.7) the original price efficiency measures, i.e., $\frac{Var(S|s)}{Var(S|p_I)}$ for market factor and $\frac{Var(v_2|\mathscr{I}_a)}{Var(v_2|p_2)}$ for individual asset.

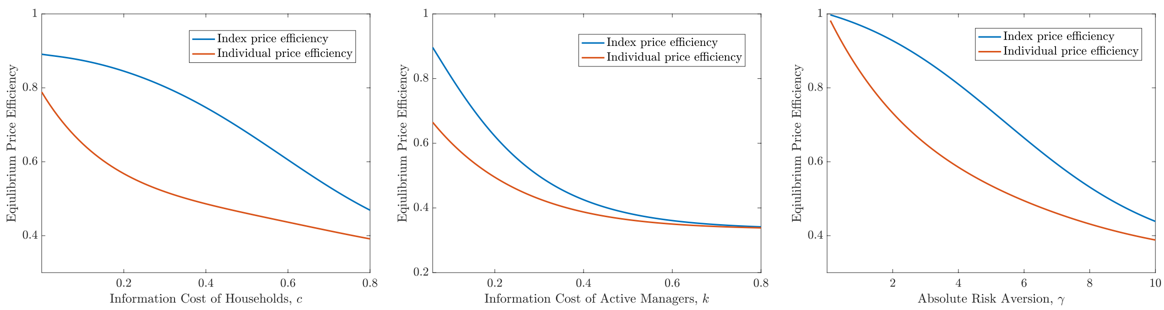


Figure 6: *Exogenous factors harming price efficiency*. As I vary the choice of $c \in [0, 0.8]$, or $k \in (0, 0.8]$, or $\gamma \in (0, 10]$, this figure shows the corresponding effect on price efficiency. Assume the individual stock has an average exposure $\beta_2 = 1$. For the other parameters, take $\gamma = 3,\ k = 0.07,\ c = 0.1$ if they are fixed variables, $\sigma_{q_I}^2 = \sigma_{q_2}^2 = \sigma_e^2 = \sigma_s^2 = 2\sigma_\varepsilon^2 = 0.2$.

[22] The argument comes directly from the results in Lemma C-1.

The model predicts that high information frictions or high risk aversion deteriorate the security-specific information and systematic information contained in prices. The outcome partly contradicts the argument in Cong and Xu (2016), in which the authors introduce factor investing exogenously. They argue that the rising factor investing harms idiosyncratic efficiency. In this model, this is true when the searching cost of households to find out manager type is tiny or when households are inclined to be risk-neutral. Then any increase in households' searching cost or in their risk aversion will give rise to factor investing, curtailing the price efficiency of assets. However, when c or γ further increases, or the fundamental information is harder for managers to obtain, there is a decreasing demand for factor investing, accompanied by decreasing price efficiency. This contradicts the outcomes in Cong and Xu (2016). Moreover, the amount of fund delegation first increases with risk aversion but then decreases, whereas price efficiency monotonically decreases. This is because price informativeness is related to the risk-appetite-adjusted delegated capital. With investors more inclined to be risk-neutral, the two effects cancel each other out.

6 Conclusion

In this paper I have developed a general equilibrium, jointly determining the households' capital allocation, their investment in finding out manager quality, the roles of fund managers, and fund fees. The outcomes contribute to understanding the segmentation of wealth under different investment vehicles, including self-investing and different types of funds.

The model also helps answer the regulatory concerns about the impact of rising factor investing on financial stability. Studies like Malikov (2019) imply that public concerns are merited as, according to his framework, the increase in indexing could be unbounded. However, in this paper, the equilibrium implies an implicit ceiling of capital in factor investing. Upon reaching this level, indexation has fully exploited its informational advantage in relation to market factor. Therefore, regulators do not need to step in to prevent factor investing from seizing 100 percent of the market.

Furthermore, the model bridges the standard REE literature with household finance. The model shows that household inequality in their wealth endowments and in their cognitive ability to detect manager quality serves as one possible explanation of the rise in factor investing. The wealth effect, in this framework, is more prominent than the cognitive ability effect. This novel result encourages future empirical testing of the links between wealth inequality and capital allocation shifts. Other avenues for future research involve looking at innovative ways of modeling cognitive ability, for instance by incorporating its impact on certainty-equivalent utility. In addition, the model claims that the falling but still not insignificant information costs, an intermediate level of household risk aversion, or high idiosyncratic risk and supply uncertainty can also give rise to factor investing.

Lastly, the model has certain implications on searching for the optimal industrial organization solution for the delegation industry. For the active industry to guarantee flows, in any equilibrium there cannot be an excessive number of closet indexers. Meanwhile managers cannot set management fees too high, as factor investing brings additional competitive pressure to the delegation industry. The model also helps understand empirical asset pricing findings related to financial intermediaries, including return volatility, co-movement, and price efficiency.

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Appendix:
The Drivers and Inhibitors of Factor Investing

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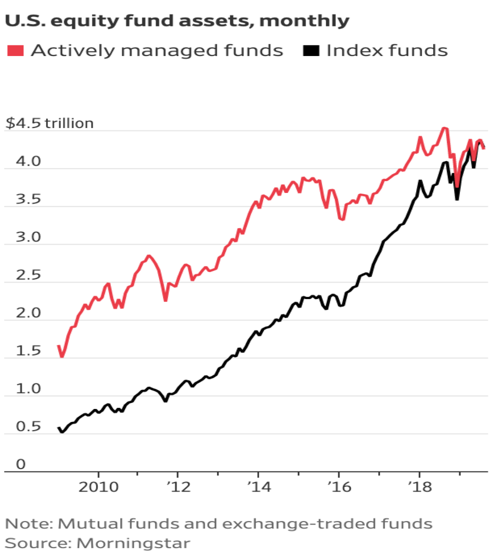
8th March, 2020

Abstract

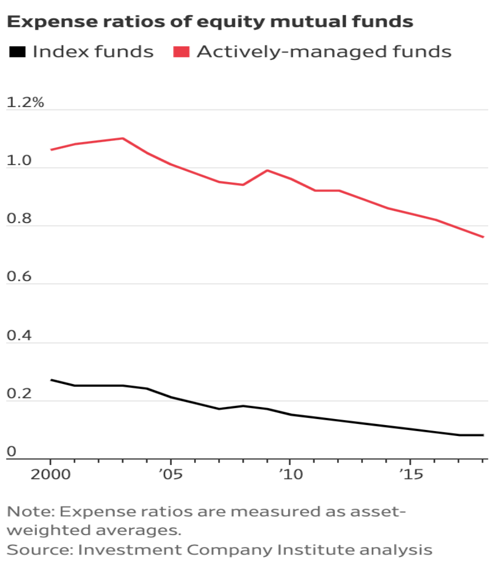
This document contains the figures and data evidence (Section A), asset orthogonalization (Section B), proofs of the results (Section C), model framework and implications with heterogeneous households (Section D), and robustness in rich asset market and discussions of fund alpha (Section E).

A Data Evidence

Among the general rise in delegated asset management over the past decade, money managers mimicking the market factor have become the new titans. In August 2019, the net asset value in indexing funds, for the first time, surpasses their actively stock-selection rivals (Panel A of Figure A-1, with \$4.27 trillion in index funds versus \$4.25 trillion in stock-picking funds as of the date). At the same time, we observe a continuous tendency of decreasing delegation fees for both types of funds, as shown in Panel B with a magnitude of roughly 0.3% for the past two decades.



A. Capital.

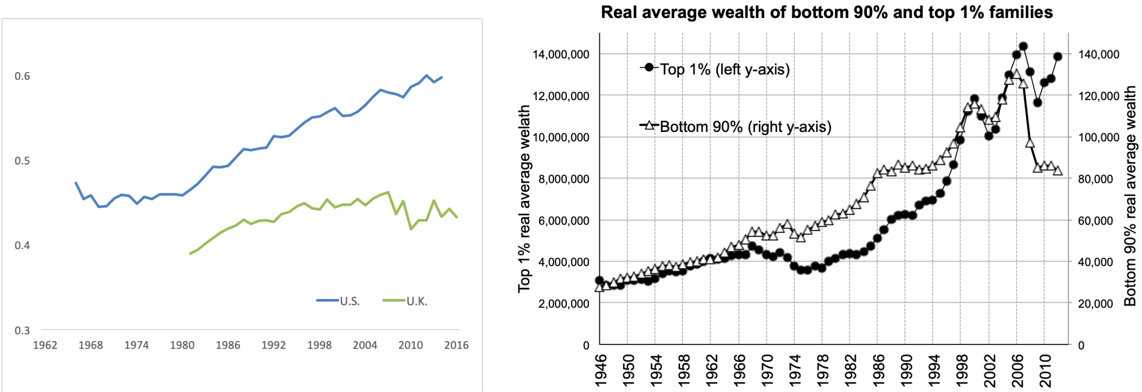


B. Fees.

Figure A-1: *U.S. delegtaion industry segmentation (Panel A) and delegation fees (Panel B).*

Many countries have experienced an increase in wealth inequality across the population. Taking the Gini coefficient as a measure for wealth inequality (a number between 0 to 1, with 1 standing for maximal inequality), over the past half century, both the U.S. and the U.K. have an upward sloping Gini plot, as shown in Panel A of

Figure A-2. Furthermore, detailed evolution of wealth gaps is provided in Saez and Zucman (2016), as shown in Panel B of Figure A-2.

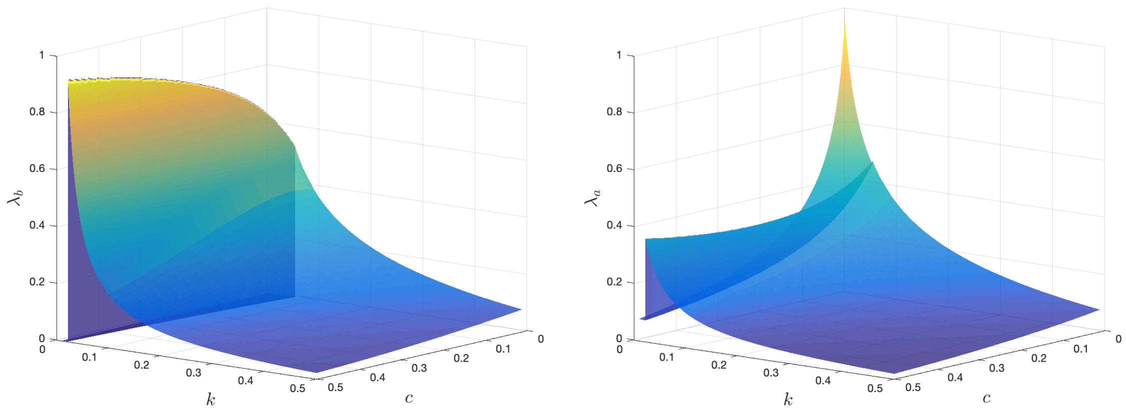


A. Gini coefficient.

B. Wealth of bottom and top U.S. families.

Figure A-2: *Time series evolution of wealth inequality.* Panel A plots the Gini coefficient of U.S. and U.K., 1962-2016, with data from World Bank. Panel B depicts the average real wealth of bottom 90% families (right y-axis) and top 1% families (left y-axis) in the U.S. from 1946 to 2012, with wealth expressed in constant 2010 U.S. dollars using the GDP deflator.

As shown in the main body of the paper, the equilibrium capital allocations of households are determined by a bunch of exogenous variables, $\{k, c, \gamma, \sigma_{q_2}^2, \sigma_e^2, \sigma_\varepsilon^2\}$. In the following example, where the exogenous variables are fixed except for the information costs (k, c), I look at the shift between equilibria.



A. Demand for factor investing.

B. Demand for fundamental investing.

Figure A-3: *Equilibrium capital allocation w.r.t. information costs.* Varying the information cost pair $(k, c) \in (0, 0.5] \times (0, 0.5]$, the figure plots the equilibrium demand for factor investing λ_b (Panel A) and fundamental investing λ_a (Panel B). For the other parameters, take $\gamma = 3$, $\sigma_{q_1}^2 = \sigma_{q_2}^2 = \sigma_e^2 = \sigma_s^2 = 2\,\sigma_\varepsilon^2 = 0.2$.

B Asset Orthogonalization

Because the risky underlying assets are correlated with each other both in terms of systematic factors and idiosyncratic factors, I apply the orthogonalization method to simplify the calculation of conditional expectation and covariance matrix. Note that there are multiple ways of orthogonalization yet in essence are identical. It is equivalent to finding an orthogonal basis, among which the elements are mutually orthogonal, of the vector space spanned by $\boldsymbol{v}$. Similar to the synthetic assets in Bond and Garcıa (2017), the asset space is re-spanned to include one market portfolio, with payoff S, and $l-1$ stock-picking assets. I apply the standard Gram-Schmidt process to find the rest orthogonal basis of the underlying asset space, given in Lemma B-1.

Lemma B-1 (**Orthogonal basis**) *One l-dimensional orthogonal basis for the asset space is*

$$
\begin{aligned}
\tilde{v}_1 &= S, \\
\tilde{v}_2 &= \mathscr{E}_2, \\
\tilde{v}_3 &= \mathscr{E}_3 - \frac{\langle \tilde{v}_2, \mathscr{E}_3 \rangle}{\langle \tilde{v}_2, \tilde{v}_2 \rangle} \tilde{v}_2 = \mathscr{E}_3 - \rho_{23} \mathscr{E}_2, \\
&\dots \\
\tilde{v}_l &= \mathscr{E}_l - \sum_{i=1}^{l-1} \frac{\langle \tilde{v}_i, \mathscr{E}_l \rangle}{\langle \tilde{v}_i, \tilde{v}_i \rangle} \tilde{v}_i,
\end{aligned}
$$

where $\langle \cdot, \cdot \rangle$ is the inner product operator.[1]

Taking $l = 3$ as an example, I show how an orthogonal basis works. The correlation between $\mathscr{E}_2$ and $\mathscr{E}_3$ is $\rho_{23} := \frac{b_1^2 - b_2^2 - b_3^2}{2 b_2 b_3}$. Then a portfolio holding of $\boldsymbol{x} = (x_1, x_2, x_3)^\intercal$ with asset payoff $(v_1, v_2, v_3)^\intercal$ is equivalent to a portfolio holding of $\tilde{\boldsymbol{x}} = (x_1 \beta_1 + x_2 \beta_2 + x_3 \beta_3, x_2 - x_1 b_2/b_1 + \rho_{23} (x_3 - x_1 b_3/b_1), x_3 - x_1 b_3/b_1)^\intercal$ with asset payoff $(S, \mathscr{E}_2, \mathscr{E}_3 - \rho_{23} \mathscr{E}_2)^\intercal$. Intuitively, in the new asset space there are three orthogonal assets—the market index and two independent stock-picking assets. The independency greatly simplifies the calculation of portfolio choice, with the detailed results of conditional expectation and variance in the next section.

I specify the noisy supply vector of the re-spanned payoff space as follows. Denote the supply of the orthogonal base $(\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_l)$ in Lemma B-1 by $\tilde{\boldsymbol{q}} = (q_I, q_2, ..., q_l)$, normally distributed and independent elementwise, with the random variables satisfying $q_I \overset{d}{\sim} \mathscr{N}(\overline{q}_I, \sigma_{q_I}^2)$, and $q_i \overset{d}{\sim} \mathscr{N}(\overline{q}_i, \sigma_{q_i}^2)$ for $i \geq 2$. Assume the original and orthogonal asset spaces are linked through a nonsingular transforming matrix Γ, i.e., $\boldsymbol{v} = \Gamma \tilde{\boldsymbol{v}}$. Then the strategy and supply vector satisfy $\boldsymbol{x} = \Gamma \tilde{\boldsymbol{x}}$ and $\boldsymbol{q} = \Gamma \tilde{\boldsymbol{q}}$.

[1] For the detailed algorithm of Gram-Schmidt process and inner product, see, for instance, Cheney and Kincaid (2009).

C Proofs

C.1 Equilibrium price

Proof of Proposition 1 In choosing the coefficients, I use subscript 'b' for the sensitivity of factor investors' signal towards price, subscript 'a' for the sensitivity of fundamental investors' signals, and subscript 'q' for the sensitivity of asset supply. I conjecture the linear forms (1) for asset prices, as functions of signals and supplies, i.e.,

$$
\begin{aligned}
p_I &= \theta_0 + \theta_b \left[(s - \bar{s}) - \theta_q \, (q_I - \bar{q}_I) \right] = \theta_0 + \theta_b \, \hat{p}_I, \\
\tilde{p}_2 &= p_2 - \beta_2 \, p_I = \phi_a \left[(e_2 - 0) - \phi_q \, (q_2 - \bar{q}_2) \right], \\
\tilde{p}_3 &= p_3 - \beta_3 \, p_I = \psi_a \left[(e_3 - 0) - \psi_q \left((q_3 - \bar{q}_3) + \psi_\rho \, (q_2 - \bar{q}_2) \right) \right],
\end{aligned}
$$

with the rest p_i following from orthogonalization. Note that in the assumed form I introduce an intermediate "auxiliary" price vector, $\hat{p}_I$, for computational convenience of conditional mean and variance. It contains the same information content as p_I. Before figuring out the parameters, I first determine the portfolio holding strategies for different investors. According to the facts from Grossman and Stiglitz (1980), the conditional mean and variance of two jointly normally distributed variables, X and Y are,

$$
\begin{aligned}
\mathbb{E}[X|Y = y] &= \mathbb{E}[X] + \frac{Cov(X,Y)}{Var(Y)} \, (y - \mathbb{E}[Y]), \\
Var(X|Y = y) &= Var(X) - \frac{[Cov(X,Y)]^2}{Var(Y)}.
\end{aligned}
\tag{C-1}
$$

(i) Factor investors and closet indexers:
They observe an imperfect signal of market factor and invest in the market portfolio. By calculation we have

$$
\mathbb{E}[S|\mathscr{I}_b] = \mathbb{E}[S|s] = \bar{s} + \frac{\sigma_s^2}{\sigma_s^2 + \sigma_\varepsilon^2} \, (s - \bar{s}) = \bar{s} + \beta_{v,s} \, (s - \bar{s}),
$$

and

$$
Var(S|\mathscr{I}_b) = Var(S|s) = \frac{\sigma_s^2 \, \sigma_\varepsilon^2}{\sigma_s^2 + \sigma_\varepsilon^2} = \beta_{v,s} \, \sigma_\varepsilon^2.
$$

Therefore, the strategy of factor investors (including closet indexers), decomposed using the orthogonal asset space, is

$$
\tilde{\boldsymbol{x}}_b = \left(\frac{\mathbb{E}[S|s] - p_I}{\gamma Var(S|s)}, 0, ..., 0 \right)^\mathsf{T}. \tag{C-2}
$$

(ii) Direct investors:
The households who choose to invest directly in the asset market, observe nothing but the public listed prices. By calculation we have

$$
\mathbb{E}[S|\mathscr{I}_h] = \mathbb{E}[S|p_I] = \mathbb{E}[S|\hat{p}_I] = \bar{s} + \frac{\sigma_s^2}{\sigma_s^2 + \sigma_\varepsilon^2 + \theta_q^2 \, \sigma_{q_I}^2} \, \hat{p}_I = \bar{s} + \beta_{v,p_I} \, \hat{p}_I,
$$

and

$$
Var(S|\mathscr{I}_h) = Var(S|p_I) = (1 - \beta_{v,p_I}) \, \sigma_s^2.
$$

Therefore, the strategy of direct investors, decomposed using the new orthogonal asset space, is

$$
\tilde{\boldsymbol{x}}_h = \left(\frac{\mathbb{E}[S|p_I] - p_I}{\gamma Var(S|p_I)}, 0, ..., 0 \right)^\mathsf{T}. \tag{C-3}
$$

(iii) Fundamental investors:
The skilled active managers observe the signals of both systematic and idiosyncratic loadings. Similarly, I define

$$
\beta_{v,e} = \frac{\sigma_e^2}{\sigma_e^2 + \sigma_\varepsilon^2}.
$$

Re-spanning the payoff space guarantees independent assets and a diagonal variance-covariance matrix:

$$\mathbb{E}\left[\begin{array}{c|c} S & \\ \mathscr{E}_2 & \\ \mathscr{E}_3-\rho_{23}\,\mathscr{E}_2 & \mathscr{I}_a \\ \dots & \end{array}\right] = \begin{bmatrix} \mathbb{E}[S|s] \\ \mathbb{E}[\mathscr{E}_2|e_2] \\ \mathbb{E}[\mathscr{E}_3-\rho_{23}\,\mathscr{E}_2|e_2,e_3] \\ \dots \end{bmatrix} = \begin{bmatrix} \bar{s}+\beta_{v,s}\,(s-\bar{s}) \\ \beta_{v,e}\,e_2 \\ \beta_{v,e}\,(e_3-\rho_{23}\,e_2) \\ \dots \end{bmatrix},$$

and

$$Var\left(\begin{array}{c|c} S & \\ \mathscr{E}_2 & \\ \mathscr{E}_3-\rho_{23}\,\mathscr{E}_2 & \mathscr{I}_a \\ \dots & \end{array}\right) = \begin{bmatrix} Var(S|s) & & & \\ & Var(\mathscr{E}_2|e_2) & & \\ & & Var(\mathscr{E}_3-\rho_{23}\,\mathscr{E}_2|e_2,e_3) & \\ & & & \dots \end{bmatrix}$$

$$= \begin{bmatrix} \beta_{v,s}\,\sigma_\varepsilon^2 & & & \\ & \beta_{v,e}\,\sigma_\varepsilon^2 & & \\ & & (1+\rho_{23}^2)\,\beta_{v,e}\,\sigma_\varepsilon^2 & \\ & & & \dots \end{bmatrix}.$$

Note that conditional variance sits on the independency of noise terms,

$$Cov(\mathscr{E}_2,\mathscr{E}_3|e_2,e_3) = Cov(\varepsilon_2,\varepsilon_3|e_2,e_3) = 0.$$

Therefore, the strategy of fundamental investors, decomposed using the orthogonal asset space, is

$$\tilde{\boldsymbol{x}}_a = \left(\frac{\mathbb{E}[S|s]-p_I}{\gamma Var(S|s)}, \frac{\mathbb{E}[\mathscr{E}_2|e_2]-\tilde{p}_2}{\gamma Var(\mathscr{E}_2|e_2)}, \frac{\mathbb{E}[\mathscr{E}_3-\rho_{23}\,\mathscr{E}_2|e_2,e_3]-\tilde{p}_3+\rho_{23}\,\tilde{p}_2}{\gamma Var(\mathscr{E}_3-\rho_{23}\,\mathscr{E}_2|e_2,e_3)},\dots\right)^{\mathsf{T}}. \tag{C-4}$$

Now I determine the coefficients for equilibrium prices. Apply the market clearing condition in the orthogonalized asset space (in essence identical to condition (2.5)), as follows,

$$\lambda_h\,\tilde{\boldsymbol{x}}_h + \big(\lambda_b+(1-\mu)\,(1-\xi)\,\lambda_a\big)\,\tilde{\boldsymbol{x}}_b + \big(\xi+\mu\,(1-\xi)\big)\,\lambda_a\,\tilde{\boldsymbol{x}}_a = \tilde{\boldsymbol{q}}, \tag{C-5}$$

where $\tilde{\boldsymbol{q}}$ is the noisy supply vector of the index and stock-picking assets, introduced in Appendix Section B. Plugging in the demands $\tilde{\boldsymbol{x}}_j$ ($j\in\{h,p,a\}$) developed above, the equations turn out to be linear in the random signals and supplies. As they have to hold for any values of q_I,q_i and s,e_i, the aggregate coefficients on these variables have to equal zero, as well as the constant term.

Solving the market clearing equation for market portfolio, with payoff S, yields

$$\begin{aligned}
\theta_q &= \frac{\gamma\sigma_\varepsilon^2}{\lambda_b+\lambda_a}, \\
\theta_b &= \frac{(\lambda_b+\lambda_a)\,\beta_{v,s}+\lambda_h\,\frac{Var(S|s)}{Var(S|p_I)}\,\beta_{v,p_I}}{(\lambda_b+\lambda_a)+\lambda_h\,\frac{Var(S|s)}{Var(S|p_I)}}, \\
\theta_0 &= \bar{s}-\frac{\bar{q}_I\,\gamma Var(S|s)}{(\lambda_b+\lambda_a)+\lambda_h\,\frac{Var(S|s)}{Var(S|p_I)}}.
\end{aligned}$$

Solving the market clearing equation for the first stock-picking asset, with payoff $\mathscr{E}_2$, yields

$$\begin{aligned}
\phi_q &= \frac{\gamma\sigma_\varepsilon^2}{\big(\xi+\mu\,(1-\xi)\big)\,\lambda_a}, \\
\phi_a &= \beta_{v,e}, \\
\bar{q}_2 &= 0.
\end{aligned}$$

Solving the market clearing equation for the second stock-picking asset, with payoff $\mathscr{E}_3 - \rho_{23}\,\mathscr{E}_2$, yields

$$
\begin{aligned}
\psi_q &= (1+\rho_{23}^2)\,\phi_q,\\
\psi_a &= \phi_a,\\
\psi_\rho &= \frac{\rho_{23}}{1+\rho_{23}^2},\\
\overline{q}_3 &= 0.
\end{aligned}
$$

Similar analysis can be applied to the rest stock-picking assets. It turns out that $\overline{q}_2 = \overline{q}_3 = \dots = \overline{q}_l = 0$. Intuitively, this implies that the expected supply of $v_1, v_2, \dots, v_l$ is proportional to their fractions in the index. Thus the weights of the stock components in index are proportional to the stocks' total market capitalization.

C.2 Price efficiency and relative utility

Landing on the results of Proposition 1 and definition (2.6), we have

$$
\eta_I = -\frac{1}{2}\log\left(1 - \frac{\sigma_\varepsilon^2\,\sigma_s^2\,\sigma_{q_I}^2/(\sigma_s^2+\sigma_\varepsilon^2)}{(\lambda_b+\lambda_a)^2/\gamma^2 + \sigma_\varepsilon^2\,\sigma_{q_I}^2}\right),
$$

where $\sigma_{q_I}^2$ is the supply uncertainty of market portfolio. One can see that η_I decreases with the amount of capital in fund delegation. Moreover, the price inefficiency does not depend on the fraction of skilled managers μ or informed households ξ. This is because the idiosyncratic information, in aggregation, does not affect the informativeness of the price of market portfolio.

Similar to the index price inefficiency (2.6), I define the price inefficiency of idiosyncratic attributes. Due to the symmetric property in choosing orthogonal basis, I take $\mathscr{E}_2$ as an example and perform necessary calculation to get,

$$
\tilde{\eta}_2 := -\frac{1}{2}\log\left(\frac{Var(\mathscr{E}_2|e_2)}{Var(\mathscr{E}_2|\tilde{p}_2)}\right) = -\frac{1}{2}\log\left(1 - \frac{\sigma_\varepsilon^2\,\sigma_e^2\,\sigma_{q_2}^2/(\sigma_e^2+\sigma_\varepsilon^2)}{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2/\gamma^2 + \sigma_\varepsilon^2\,\sigma_{q_2}^2}\right) \in (0,\infty), \tag{C-6}
$$

where $\sigma_{q_2}^2$ is the supply uncertainty of the stock-picking asset with payoff $\mathscr{E}_2$. This helps investigate the price inefficiency of the individual asset. For stock 2, its price inefficiency is defined to be and equal to

$$
\eta_2 := -\frac{1}{2}\log\left(\frac{Var(v_2|\mathscr{I}_a)}{Var(v_2|\boldsymbol{p})}\right) = -\frac{1}{2}\log\left(\frac{\beta_2^2\,\frac{\sigma_s^2}{\sigma_s^2+\sigma_\varepsilon^2} + \frac{\sigma_e^2}{\sigma_e^2+\sigma_\varepsilon^2}}{\beta_2^2\,\frac{\sigma_s^2}{\sigma_s^2+\sigma_\varepsilon^2}\,\frac{(\lambda_b+\lambda_a)^2+\gamma^2\,\sigma_\varepsilon^2\,\sigma_{q_I}^2}{(\lambda_b+\lambda_a)^2+\gamma^2\,\sigma_\varepsilon^4\,\sigma_{q_I}^2} + \frac{\sigma_e^2}{\sigma_e^2+\sigma_\varepsilon^2}\,\frac{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2+\gamma^2\,\sigma_\varepsilon^2\,\sigma_{q_2}^2}{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2+\gamma^2\,\sigma_\varepsilon^4\,\sigma_{q_2}^2}}\right).
$$

Recall the notations $\beta_{v,s} = \frac{\sigma_s^2}{\sigma_s^2+\sigma_\varepsilon^2}$ and $\beta_{v,e} = \frac{\sigma_e^2}{\sigma_e^2+\sigma_\varepsilon^2}$. The three inefficiency expressions can be linked together in a following way,

$$
e^{-2\,\eta_2} = \frac{Var(v_2|\mathscr{I}_a)}{Var(v_2|\boldsymbol{p})} = \frac{\beta_2\,Var(S|s) + Var(\mathscr{E}_2|e_2)}{\beta_2\,Var(S|p_I) + Var(\mathscr{E}_2|\tilde{p}_2)} = \frac{\beta_2\,\beta_{v,s} + \beta_{v,e}}{\beta_2\,\beta_{v,s}\,e^{2\,\eta_I} + \beta_{v,e}\,e^{2\,\tilde{\eta}_2}}. \tag{C-7}
$$

The price efficiency can be linked to the relative utility generated by fund performance, in the following way. Take $l=3$ as an example.

Lemma C-1 (**Relative utility**) *The excess performance of delegation over household uninformed investing, represented using the relative utility, can be linked to the price efficiency:*

$$
\gamma(u_b - u_h) = \eta_I,
$$

$$
\gamma(u_a - u_h) = \eta_I + \frac{1}{2}\log\left(1 + \frac{\gamma^2\,\beta_{v,e}\,\sigma_\varepsilon^2\,\sigma_{q_2}^2}{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2}\right) + \frac{1}{2}\log\left(\frac{1+\rho_{23}^2\,(2\,\beta_{v,e}-1)}{1+\rho_{23}^2} + \frac{\gamma^2\,(1+\rho_{23}^2)\,\beta_{v,e}\,\sigma_\varepsilon^2}{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2}\,\sigma_{q_3}^2\right).
$$

PROOF: First note that

$$e^{-\gamma u_j} = \mathbb{E}\left[\exp\left\{-\frac{1}{2}\left(\mathbb{E}[\boldsymbol{v}|\mathscr{I}_j]-\boldsymbol{p}\right)^{\intercal}\boldsymbol{\Sigma}_j^{-1}\left(\mathbb{E}[\boldsymbol{v}|\mathscr{I}_j]-\boldsymbol{p}\right)\right\}\right],$$

where $\boldsymbol{\Sigma}_j$ is the conditional variance-covariance matrix of $\boldsymbol{v}$, and $\mathbb{E}[\boldsymbol{v}|\mathscr{I}_j]$ is the conditional mean, for any investor type $j \in \{h,p,a\}$. After orthogonalization, we have for any type $j \in \{h,p\}$,

$$e^{-\gamma u_j} = \mathbb{E}\left[\exp\left\{-\frac{1}{2}\frac{(\mathbb{E}[S|\mathscr{I}_j]-p_I)^2}{Var(S|\mathscr{I}_j)}\right\}\right];$$

and for fundamental investors ($j=a$),

$$e^{-\gamma u_a} = \mathbb{E}\left[\exp\left\{-\frac{1}{2}\left(\frac{(\mathbb{E}[S|\mathscr{I}_a]-p_I)^2}{Var(S|\mathscr{I}_a)}+\frac{(\mathbb{E}[\mathscr{E}_2|\mathscr{I}_a]-\tilde{p}_2)^2}{Var(\mathscr{E}_2|\mathscr{I}_a)}+\frac{(\mathbb{E}[\mathscr{E}_3-\rho_{23}\,\mathscr{E}_2|\mathscr{I}_a]-\tilde{p}_3+\rho_{23}\,\tilde{p}_2)^2}{Var(\mathscr{E}_3-\rho_{23}\,\mathscr{E}_2|\mathscr{I}_a)}\right)\right\}\right].$$

To complete the proof, I need the following facts. First, for independently normally distributed variables $z_i \overset{d}{\sim} \mathscr{N}(\mu_i, V_i)$, $i=1,2,3$, we have

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\left(z_1^2+z_2^2+z_3^2\right)\right\}\right] = \prod_{i=1}^{3}(1+V_i)^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\frac{\mu_i^2}{1+V_i}\right\},$$

which comes form the moment-generating function of multi-dimensional non-central chi-square distribution. Second, according to the facts (C-1) listed in the proof of Proposition 1, we have

$$Var(\mathbb{E}[X|Y]) = \frac{[Cov(X,Y)]^2}{Var(Y)} = Var(X) - Var(X|Y), \tag{C-8}$$

which in fact comes from the law of total variance: the unconditional variance of a posterior belief equals the variance of the prior minus the posterior variance.

Then, performing necessary calculations yields

$$\begin{aligned}
u_a &= \frac{1}{2\gamma}\left[\log\left(\frac{Var(S-p_I)}{Var(S|s)}\right)+\log\left(\frac{Var(\mathscr{E}_2-\tilde{p}_2)}{Var(\mathscr{E}_2|e_2)}\right)+\log\left(\frac{Var(\mathscr{E}_3-\rho_{23}\,\mathscr{E}_2-\tilde{p}_3+\rho_{23}\,\tilde{p}_2)}{Var(\mathscr{E}_3-\rho_{23}\,\mathscr{E}_2|e_2,e_3)}\right)+\frac{(\bar{s}-\theta_0)^2}{Var(S-p_I)}\right],\\
u_b &= \frac{1}{2\gamma}\left[\log\left(\frac{Var(S-p_I)}{Var(S|s)}\right)+\frac{(\bar{s}-\theta_0)^2}{Var(S-p_I)}\right],\\
u_h &= \frac{1}{2\gamma}\left[\log\left(\frac{Var(S-p_I)}{Var(S|p_I)}\right)+\frac{(\bar{s}-\theta_0)^2}{Var(S-p_I)}\right].
\end{aligned}$$

Note that the last term in the certainty-equivalent wealth, $\frac{1}{2\gamma}\frac{(\bar{s}-\theta_0)^2}{Var(S-p_I)}$, stands for the monetary utility of investing in market portfolio through unconditional mean-variance strategies.

Given the definition of index price inefficiency, equation (2.6), it is straightforward that

$$\gamma(u_b-u_h) = \eta_I.$$

Now I investigate the excess performance of active funds. Based on the results from Proposition 1, one can see that

$$\gamma(u_a-u_h) = \eta_I + \frac{1}{2}\log\left(\frac{Var(\mathscr{E}_2-\tilde{p}_2)}{\beta_{v,e}\,\sigma_\varepsilon^2}\right)+\frac{1}{2}\log\left(\frac{Var(\mathscr{E}_3-\rho_{23}\,\mathscr{E}_2-\tilde{p}_3+\rho_{23}\,\tilde{p}_2)}{(1+\rho_{23}^2)\,\beta_{v,e}\,\sigma_\varepsilon^2}\right). \tag{C-9}$$

Performing necessary calculation yields the unconditional variance of the stock-picking asset returns,

$$Var(\mathscr{E}_2-\tilde{p}_2) = (1-\phi_a)^2\,\sigma_e^2+\phi_a^2\,\sigma_\varepsilon^2+\phi_a^2\,\phi_q^2\,\sigma_{q_2}^2 = \beta_{v,e}\,\sigma_\varepsilon^2\left(1+\frac{\gamma^2\,\beta_{v,e}\,\sigma_\varepsilon^2}{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2}\,\sigma_{q_2}^2\right), \tag{C-10}$$

and

$$
\begin{aligned}
Var(\mathscr{E}_3 - \rho_{23}\,\mathscr{E}_2 - \tilde{p}_3 + \rho_{23}\,\tilde{p}_2) &= (1-\rho_{23}^2)\,(1-\phi_a)^2\,\sigma_e^2 + (1+\rho_{23}^2)\,\phi_a^2\,\sigma_\varepsilon^2 + (1+\rho_{23}^2)^2\,\phi_a^2\,\phi_q^2\,\sigma_{q_3}^2 \\
&= \beta_{v,e}\,\sigma_\varepsilon^2\left(1 + \rho_{23}^2\,(2\,\beta_{v,e}-1) + \frac{\gamma^2\,(1+\rho_{23}^2)^2\,\beta_{v,e}\,\sigma_\varepsilon^2}{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2}\,\sigma_{q_3}^2\right).
\end{aligned}
\tag{C-11}
$$

Thus equation (C-9) becomes

$$
\gamma(u_a-u_h) = \eta_I + \frac{1}{2}\,\log\Big(1+\frac{\gamma^2\,\beta_{v,e}\,\sigma_\varepsilon^2}{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2}\,\sigma_{q_2}^2\Big) + \frac{1}{2}\,\log\Big(\frac{1+\rho_{23}^2\,(2\,\beta_{v,e}-1)}{1+\rho_{23}^2} + \frac{\gamma^2\,(1+\rho_{23}^2)\,\beta_{v,e}\,\sigma_\varepsilon^2}{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2}\,\sigma_{q_3}^2\Big).
$$

□

Let us take a look at the extreme scenario where fund managers have no inside information, mathematically, $\sigma_\varepsilon^2 = +\infty$. By definition (2.6) it is not hard to figure out that $\eta_I = \tilde{\eta}_2 = 0$, in other words, all the available information is contained in the price. Obviously, factor investing can do no better than direct investing utility-wise, $u_b = u_h$. However, due to the investment restrictions that, both factor funds and households themselves can only access the market portfolio in aggregation, for "hedging" purpose the active managers could still earn some market share, even in the case where they hold no advanced information. This can be seen from the relative utility result in Lemma C-1.

The main interests are in the non-extreme cases, where the signal is neither perfect nor completely useless, i.e., $\sigma_\varepsilon^2 \in (0,\infty)$. A natural concern is how does *information accuracy* affect price efficiency. Intuitively, a smaller σ_ε^2 means a more accurate signal. The effect of signal precision depends on how active the assets are traded. A relatively *liquid* market is featured by more frequent trading behavior, thus, in my model is characterized by a relatively larger $\sigma_{q_i}^2$ and $\sigma_{q_I}^2$. The relation between information accuracy and price efficiency is stated below.

Lemma C-2 (Information accuracy) *(i) If market is relatively liquid, i.e., $\sigma_s^2 \le \theta_q^2\,\sigma_{q_I}^2$ and $\sigma_e^2 \le \phi_q^2\,\sigma_{q_2}^2$, then an increase in information accuracy increases the price efficiency of index and stock-picking assets;*

(ii) If market is relatively illiquid, i.e., $\sigma_s^2 > \theta_q^2\,\sigma_{q_I}^2$ and $\sigma_e^2 > \phi_q^2\,\sigma_{q_2}^2$, then an increase in information accuracy weakly decreases the price efficiency of index and stock-picking assets.

PROOF: Recall the expressions of η_I, $\tilde{\eta}_2$, and η_2. The higher the numbers, the less efficient the prices are. Now I investigate the relation between σ_ε^2 and them. For the market index, for instance, consider the following derivative

$$
\mathscr{D}\Big[\frac{\sigma_\varepsilon^2\,\sigma_s^2\,\sigma_{q_I}^2/(\sigma_s^2+\sigma_\varepsilon^2)}{(\lambda_b+\lambda_a)^2/\gamma^2+\sigma_\varepsilon^2\,\sigma_{q_I}^2},\,\sigma_\varepsilon^2\Big].
$$

Performing necessary calculation, it is positive if and only if

$$
\sigma_s^2 > \theta_q^2\,\sigma_{q_I}^2. \tag{C-12}
$$

That is to say, if condition (C-12) is satisfied, then an increase in σ_ε^2 increases η_I thus deteriorates the price efficiency. Performing similar analysis on $\tilde{\eta}_2$, we see the results in the lemma hold. □

Another controversial topic to practitioners and academia is that, how does the increase in factor investing affect the price informativeness of relatively liquid vs. illiquid stocks? A more actively traded stock, in my framework, is mapped into the one with a relatively larger $\sigma_{q_i}^2$. Then Lemma C-3 gives answer to the question.

Lemma C-3 (Factor investing on liquid vs. illiquid assets) *Holding the total fraction of wealth under delegation fixed,*

(i) The rising factor investing has larger impact on the price informativeness of illiquid assets than on the liquid ones;

(ii) *If an asset is extremely illiquid in terms of its idiosyncratic part, then the price is almost perfectly efficient regardless of the change of capital allocation.*

PROOF: I compare two assets with different liquidity to show the results. Counting on definition (C-6), we have

$$\tilde{\eta}_2 := -\frac{1}{2}\log\Big(\frac{Var(\mathscr{E}_2|e_2)}{Var(\mathscr{E}_2|\tilde{p}_2)}\Big), \quad \tilde{\eta}_3 := -\frac{1}{2}\log\Big(\frac{Var(\mathscr{E}_3|e_3)}{Var(\mathscr{E}_3|\tilde{p}_3)}\Big).$$

To get the price efficiency of idiosyncratic loading $\mathscr{E}_3$, I simply resort to another orthogonal basis—$(S,\mathscr{E}_3,\mathscr{E}_2-\rho_{23}\,\mathscr{E}_3)$, where ρ_{23} is still the correlation between $\mathscr{E}_2$ and $\mathscr{E}_3$. The second element of the new basis, exactly what is needed, is now isolated. I also link the two basis with a transforming matrix:

$$\begin{bmatrix} S \\ \mathscr{E}_3 \\ \mathscr{E}_2-\rho_{23}\,\mathscr{E}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho_{23} & 1 \\ 0 & 1-\rho_{23}^2 & -\rho_{23} \end{bmatrix} \begin{bmatrix} S \\ \mathscr{E}_2 \\ \mathscr{E}_3-\rho_{23}\,\mathscr{E}_2 \end{bmatrix}.$$

Thus the noise of the supply of new basis with re-spanned payoff, $(S,\mathscr{E}_3,\mathscr{E}_2-\rho_{23}\,\mathscr{E}_3)^\intercal$, can be expressed using the old ones as $(\sigma_{q_l}^2,\rho_{23}^2\,\sigma_{q_2}^2+\sigma_{q_3}^2,(1-\rho_{23}^2)^2\,\sigma_{q_2}^2+\rho_{23}^2\,\sigma_{q_3}^2)^\intercal$.

Without loss of generality, assume $\mathscr{E}_2$ is more liquid than $\mathscr{E}_3$, i.e., $\sigma_{q_2}^2 > \rho_{23}^2\,\sigma_{q_2}^2+\sigma_{q_3}^2$. Following the above results, the price inefficiency of the two assets are

$$\begin{cases} \tilde{\eta}_2 &= -\frac{1}{2}\log\left(1-\frac{\sigma_\varepsilon^2\,\sigma_e^2/(\sigma_e^2+\sigma_\varepsilon^2)}{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2/\gamma^2\,\sigma_{q_2}^2+\sigma_\varepsilon^2}\right), \\ \tilde{\eta}_3 &= -\frac{1}{2}\log\left(1-\frac{\sigma_\varepsilon^2\,\sigma_e^2/(\sigma_e^2+\sigma_\varepsilon^2)}{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2/\gamma^2\,(\rho_{23}^2\,\sigma_{q_2}^2+\sigma_{q_3}^2)+\sigma_\varepsilon^2}\right). \end{cases}$$

Comparing the coefficients before λ_a in the expressions of price inefficiency yields the results.

If an asset, say $\mathscr{E}_2$, is extremely illiquid, i.e., $\sigma_{q_2}^2 \to 0$. Then from the expression one can conclude that $\tilde{\eta}_2 \to 0$, intuitively, the price is always perfectly informative. □

C.3 Equilibria and stylized facts

Proof of Lemma 1 The results follow directly from the relative utility outcome in Lemma C-1. Also in the proof of interior equilibrium I will show that to guarantee the inflows of active industry, μ cannot be too small. As long as $\mu > 0$, in any equilibrium, before fees we have

$$\mu\,u_a+(1-\mu)\,u_b > u_b.$$

Proof of Proposition 2 I mainly use condition (3.1) to determine λ_b. Rearranging the equation yields

$$u_a-\alpha_a-c = u_a-(u_b-u_h+k)-c = u_h,$$

which, after simplification, implies

$$\gamma(u_a-u_b) = \gamma(k+c). \tag{C-13}$$

Counting on the relative utility result from Lemma C-1, with two risky assets (one market portfolio and one stock-picking asset) we have

$$\gamma(u_a-u_b) = \frac{1}{2}\log\Big(1+\frac{\gamma^2\,\beta_{v,e}\,\sigma_\varepsilon^2}{(\xi+\mu\,(1-\xi))^2\,\lambda_a^2}\,\sigma_{q_2}^2\Big). \tag{C-14}$$

Note that equation (3.6), (3.9), and (3.10) imply the following relationship:

$$\big(\xi+\mu\,(1-\xi)\big)\,\lambda_a=\frac{k\,\gamma\mu+\eta_I\,\mu}{k\,\gamma\mu+\eta_I}\,\lambda_a=\mu\,\lambda_b.$$

Counting on this result and equating expression (C-13) and (C-14), one can solve for the equilibrium fraction of capital in factor investing, as follows,

$$\lambda_b=\frac{\gamma\sigma_\varepsilon\,\sigma_{q_2}}{\mu}\sqrt{\frac{\beta_{v,e}}{e^{2\,\gamma(k+c)}-1}}. \tag{C-15}$$

Because μ is uniquely determined given information costs k and c, λ_b is uniquely determined.

Then let us take a look at λ_a. The total capital in funds is related to the index price efficiency. Thus from definition (2.6), one get

$$\lambda_a+\lambda_b=\gamma\sigma_\varepsilon\,\sigma_{q_I}\sqrt{\frac{\beta_{v,s}}{1-e^{-2\,\eta_I}}-1}.$$

Given the ratio between demand for factor and fundamental investing (3.10), the following expression holds:

$$\frac{\lambda_a+\lambda_b}{\lambda_b}=\frac{(\mu+1)\,k\,\gamma+2\,\eta_I}{k\,\gamma+\eta_I}.$$

As we already know the equilibrium demand for factor investing λ_b, I first combine the results to solve η_I:

$$\frac{\lambda_a+\lambda_b}{\mu\,\lambda_b}=\frac{(\mu+1)\,k\,\gamma+2\,\eta_I}{\mu\,(k\,\gamma+\eta_I)}=\frac{\sigma_{q_I}}{\sigma_{q_2}}\,\frac{\sqrt{\frac{\beta_{v,s}}{1-e^{-2\,\eta_I}}-1}}{\sqrt{\frac{\beta_{v,e}}{e^{2\,\gamma(k+c)}-1}}}. \tag{C-16}$$

Then equilibrium active wealth λ_a follows.

In equation (C-16), the LHS is monotonically increasing in η_I, and the RHS is monotonically decreasing in η_I. In addition, both sides vary within a positive support region. Hence, once there exists an η_I equating the two sides, it must be unique. One can regard it as the intersection point of two functions of η_I. The case where such η_I does not exist would be discussed in the boundary equilibria cases.

Proof of Proposition 3 Given $\lambda_b+\lambda_a=1$, the model directly determines the efficiency of index price:

$$\eta_I=-\frac{1}{2}\,\log\left(1-\frac{\gamma^2\,\sigma_\varepsilon^2\,\sigma_s^2\,\sigma_{q_I}^2/(\sigma_s^2+\sigma_\varepsilon^2)}{1+\gamma^2\,\sigma_\varepsilon^2\,\sigma_{q_I}^2}\right).$$

From the ratio

$$\frac{\lambda_b}{\lambda_a}=\frac{\lambda_b}{1-\lambda_b}=\frac{\alpha_b+k}{\alpha_b+\mu\,k},$$

the factor fund fees α_b is uniquely determined, then by relation (3.4) α_a as well:

$$\alpha_b=\frac{k\big[1-(1+\mu)\,\lambda_b\big]}{2\,\lambda_b-1},\qquad \alpha_a=\frac{k\,(1-\mu)\,\lambda_b}{2\,\lambda_b-1}.$$

From condition (3.7) and (3.8) I can express some key parameters as functions of α_b:

$$\xi=\frac{\mu\,k}{\mu\,k+\alpha_b},\qquad \mu=\frac{e^{\gamma k}-1}{e^{\gamma k}-e^{-\gamma c}}\in\big[1-e^{-\gamma k},1\big].$$

One can determine demand for factor investing, λ_b, similarly using relative utility results as in the proof of Proposition 2. Then $\lambda_a=1-\lambda_b$. Up till now the equilibrium has been solved analytically.

Now I take a look at how the active delegation fees change when demand for factor investing λ_b varies. Rearranging the expression of α_a in the proposition yields

$$\alpha_a=\frac{k\,(1-\mu)}{2-1/\lambda_b}.$$

Thus an increase in λ_b, possibly brought up by the exogenous costs or supply shocks, most of the time is accompanied by active funds lowering their fees α_a on average.

Proof of Lemma 2 For the delegation dominating equilibrium to hold, we must have $\alpha_b \geq 0$ and $\xi \leq 1$. Thus the following conditions must hold,

$$\begin{cases} 1 - \lambda_b - \mu\, \lambda_b \geq 0, \\ 2\, \lambda_b - 1 > 0, \end{cases}$$

which yields the necessary conditions in the lemma.

Proof of Proposition 4 The coexistence of factor and active fund managers, condition (3.8), together with $\lambda_b = 0$, implies that all the households would like to incur a cost to become informed towards manager quality, i.e., $\xi = 1$. Thus applying condition (3.7) we get

$$\alpha_a = k, \qquad \alpha_b = \alpha_a - k = 0,$$

in other words, the funds are driven to no profit boundary. With zero delegation fees, one can see that $u_b = u_h$, in which case the price of market portfolio is fully informative ($\eta_I = 0$) that factor investing has no advantage over direct investing. This is also intuitive that a large amount of informed trading in market factor brings up the information embedded into the price and squeezes the profiting space for factor funds.

The coexistence of (i) informed and uninformed households and (ii) direct investing and fundamental investing implies that

$$\gamma\,(u_a - u_b) = \log\Big(\frac{e^{\gamma c} - \mu}{1 - \mu}\Big) = \gamma\,(k + c).$$

Thus μ is determined as in (3.6). Combining the above results and recalling the relative utility relation, I determine λ_a through

$$\frac{1}{2}\,\log\Big(1 + \frac{\gamma^2\, \beta_{v,e}\, \sigma_\varepsilon^2}{\lambda_a^2}\, \sigma_{q_2}^2\Big) = \gamma\,(k + c),$$

yielding the equilibrium capital allocated to active funds given in the proposition.

Proof of Lemma 3 The feasibility of the fundamental-dominating equilibrium within the region specified by condition (3.14) is not hard to justify. In the proof I mainly show the continuity at the switching point.

Upon reaching the peak level of factor investing, the delegation dominating equilibrium implies that

$$\lambda_b = \frac{\gamma\, \sigma_{q_2}}{\mu}\, \sqrt{\frac{1}{\big(\frac{1}{\sigma_e^2} + \frac{1}{\sigma_\varepsilon^2}\big)\,\big(e^{2\,\gamma\,(k+c)} - 1\big)}} = \frac{1}{\mu + 1},$$

and

$$\lambda_a = 1 - \lambda_b = \frac{\mu}{1 + \mu}.$$

Based on the results in Proposition 3, we shall have

$$\alpha_b = \frac{k\big[1 - (1 + \mu)\, \lambda_b\big]}{2\, \lambda_b - 1} = 0, \qquad \alpha_a = \frac{k\,(1 - \mu)\, \lambda_b}{2\, \lambda_b - 1} = \frac{k\,(1 - \mu)}{1 - \mu} = k;$$

and

$$\xi = \frac{\big[2\, \lambda_b - 1\big]\, \mu}{\big[1 - \lambda_b\big]\,(1 - \mu)} = \frac{\big[2 - (\mu + 1)\big]\, \mu}{(\mu + 1 - 1)\,(1 - \mu)} = 1.$$

The equilibrium amount of skilled active managers, μ, remains unchanged as in (3.6) and is determined by the information frictions and risk appetites. Condition (3.8) is degenerate in this case and is automatically satisfied. At this upper bound, factor investing has reached its maximal market share, with the marginal value added by the factor investors before fees approaching zero, mathematically, $u_b - u_h \to 0$.

This, with factor investing driven to non-profitable boundary, is connected to the fundamental-dominating equilibrium, in which case alongside this boundary line we have

$$\lambda_a = \gamma \sigma_{q_2} \sqrt{\frac{1}{\left(\frac{1}{\sigma_e^2} + \frac{1}{\sigma_\varepsilon^2}\right)\left(e^{2\gamma(k+c)} - 1\right)}} = \frac{\mu}{\mu+1},$$

where the first equality comes from the results in Proposition 4 and the second equality comes from the boundary condition implied by (3.13) and (3.14). Therefore, I show that alongside the boundary, any choice of (λ_b, λ_h) satisfying $\lambda_b + \lambda_h = 1/(1+\mu)$ forms an equilibrium. This is because households treat direct investing and factor investing equally. I can now conclude that the "switching" between equilibria at the boundary is always upper-semicontinuous.

Proof of Proposition 5 From equation (C-15), the demand for factor investing in the interior equilibrium is

$$\lambda_b = \frac{\gamma \sigma_\varepsilon \sigma_{q_2}}{\mu} \sqrt{\frac{\beta_{v,e}}{e^{2\gamma(k+c)} - 1}},$$

where μ is a function of information costs and risk appetite, given by expression (3.6).

(i) Taking derivative of λ_b with respect to household searching cost c yields

$$\frac{\partial \lambda_b}{\partial c} = -\frac{\gamma e^{-\gamma c}\left(e^{2\gamma(k+c)} - e^{\gamma(k+c)} - 1\right)}{\left(e^{\gamma k} - e^{-\gamma c}\right)\left(1 + e^{\gamma(k+c)}\right)} \lambda_b,$$

which is positive if $c < \frac{1}{\gamma} \log\left(\frac{(1+\sqrt{5})\, e^{-\gamma k}}{2}\right)$ and negative otherwise. Thus λ_b is of a humped shape with respect to searching cost c.

(ii) Taking derivative of λ_b with respect to managers' information cost k yields

$$\frac{\partial \lambda_b}{\partial k} = -\frac{\gamma e^{-\gamma(k-c)}\left(e^{2\gamma(k+c)} - e^{\gamma(k+c)} + e^{\gamma c} - 1\right)}{\left(e^{\gamma k} - 1\right)\left(e^{\gamma k} - e^{-\gamma c}\right)\left(1 + e^{\gamma(k+c)}\right)} \lambda_b < 0.$$

Hence, λ_b is monotonically decreasing in k.

(iii) Taking derivative of λ_b with respect to level of risk aversion γ yields

$$\begin{aligned}\frac{\partial \lambda_b}{\partial \gamma} = -\frac{e^{-\gamma c}\, \sigma_{q_2}\, \sigma_\varepsilon}{\sqrt{2}\left(e^{\gamma k} - 1\right)^2 \left(1 + e^{\gamma(k+c)}\right)} \sqrt{\beta_{v,e}\left(\frac{e^{\gamma(k+c)} + e^{-\gamma(k+c)}}{e^{\gamma(k+c)} - e^{-\gamma(k+c)}}\right)} \Big(1 - \gamma c - e^{\gamma(k+c)}\, \gamma(k+c) \\ + e^{\gamma(2k+c)}\, \gamma(k+c) + e^{2\gamma(k+c)}\,(c\gamma - 1) + \left(e^{\gamma(2c+3k)} - e^{\gamma k}\right)(1 - c\gamma - k\gamma)\Big).\end{aligned}$$

Hence, how λ_b varies with respect to γ depends on the zeros of expression

$$1 - \gamma c - e^{\gamma(k+c)}\, \gamma(k+c) + e^{\gamma(2k+c)}\, \gamma(k+c) + e^{2\gamma(k+c)}\,(c\gamma - 1) + \left(e^{\gamma(2c+3k)} - e^{\gamma k}\right)(1 - c\gamma - k\gamma).$$

(iv) Taking derivative of λ_b with respect to individual supply uncertainty $\sigma_{q_2}^2$ yields

$$\frac{\partial \lambda_b}{\partial \sigma_{q_2}^2} = \frac{e^{-\gamma k} - e^{-\gamma c}}{\left(e^{\gamma k} - 1\right)} \gamma \sigma_\varepsilon \sqrt{\frac{\beta_{v,e}}{e^{2\gamma(k+c)} - 1}} > 0.$$

Similarly by taking derivative of λ_b with respect to idiosyncratic risk σ_e^2, one can see that λ_b is monotonically increasing in $\sigma_{q_2}^2$ and σ_e^2.

(v) For this part I only have numerical results, as discussed in Appendix Section D.

Proof of Lemma 8

(i) Taking λ_h as given, I mainly investigate the influence on price terms brought by the change of $r_{f-f} = \lambda_b/\lambda_a$. Because the total demand is normalized to be a unit, we have $\lambda_a = (1-\lambda_h)/(1+r_{f-f})$. Thus ϕ_q

can be rewritten as

$$\phi_q = \frac{\gamma \sigma_\varepsilon^2 \left(1 + r_{f-f}\right)}{\left(\xi + \mu \left(1 - \xi\right)\right) \left(1 - \lambda_h\right)}. \tag{C-17}$$

If the effect on r_{f-f} dominates the change in direct investing, λ_h, then a change of demand for factor investing affects ϕ_q directly.

(a) Following the results in Proposition 1, the expected price of market index, after rearranging, is

$$\mathbb{E}[p_I] = \theta_0 = \bar{s} - \frac{\bar{q}_I \gamma Var(S|s)}{\left(\lambda_b + \lambda_a\right) \left(1 - \frac{Var(S|s)}{Var(S|p_I)}\right) + \frac{Var(S|s)}{Var(S|p_I)}}.$$

Note that $\frac{Var(S|s)}{Var(S|p_I)} \in (0,1)$. $Var(S|p_I)$ is a function of θ_q. Thus the expected price is only affected by the total amount of capital in funds. So does the expected price of individual asset if it is positively exposed to the aggregate market risk, because

$$\mathbb{E}[p_i] = \beta_i \mathbb{E}[p_I], \quad \text{for } i = 1, 2, ..., l.$$

The variance of index price is,

$$Var(p_I) = \theta_b^2 \sigma_s^2 + \theta_b^2 \theta_q^2 \sigma_{q_I}^2,$$

which also remains unchanged provided that $\lambda_b + \lambda_a$ is unaffected. The variance of stock-picking asset price, for instance the first stock-picking asset with payoff $\mathscr{E}_2$, is

$$Var(\tilde{p}_2) = \phi_a^2 \sigma_e^2 + \phi_a^2 \phi_q^2 \sigma_{q_2}^2.$$

If a rise in λ_b mainly increases r_{f-f} (decreases λ_a), then ϕ_q is larger according to equation (C-17), with the rest coefficients remaining unaffected. Hence $Var(\tilde{p}_2)$ increases. The effect of r_{f-f} on variance is approximately quadratic.

I call a market with more indexation a more "concentrated" market. Note that when an idiosyncratic shock occurs ($\sigma_{q_2}^2$, measuring the size of the shock, varies), a more concentrated industry is more exposed, due to the increased sensitivity towards the shock implied by the variance analysis.

(b) I calculate the covariance of asset-specific components using the definition, i.e.,

$$Cov(\tilde{p}_2, \tilde{p}_3) = \phi_a^2 Cov(e_2, e_3) + \rho_{23} \phi_a^2 \phi_q^2 \sigma_{q_2}^2.$$

Because we have $Cov(e_2, e_3) = Cov(\mathscr{E}_2 + \varepsilon_2, \mathscr{E}_3 + \varepsilon_3) = Cov(\mathscr{E}_2, \mathscr{E}_3) = \rho_{23} \sigma_e^2$, the covariance is of the same sign as ρ_{23}. The covariance moves in a similar fashion as the variance of individual asset's price. The expansion of factor delegation, in other words, the shrink of active delegation, drives up ϕ_q, thus the covariance between the prices of asset-idiosyncratic loadings rises (in the absolute value sense).[2] The effect of r_{f-f} on covariance is approximately quadratic.

(c) The conclusion comes directly from result (C-10) and (C-11) in Lemma C-1.

(ii) Internalizing λ_h, we have $\lambda_h = r_{n-f}/(1 + r_{n-f})$ and $\lambda_b + \lambda_a = 1/(1 + r_{n-f})$.

(a) Following the results in Proposition 1, we know that $\mathbb{E}[p_I] = \theta_0$. Differentiating it with respect to r_{n-f} yields

$$\frac{\partial \theta_0}{\partial r_{n-f}} = -\frac{\bar{q}_I \gamma^3 \sigma_{q_I}^2 \sigma_s^4 \sigma_\varepsilon^4 \left(1 + 2 r_{n-f} + \gamma^2 \left(1 + r_{n-f}\right)^2 \sigma_{q_I}^2 \sigma_\varepsilon^2\right)}{\left(\sigma_\varepsilon^2 + \gamma^2 \left(1 + r_{n-f}\right)^2 \sigma_{q_I}^2 \sigma_\varepsilon^4 + \sigma_s^2 \left(1 + \gamma^2 \left(1 + r_{n-f}\right) \sigma_{q_I}^2 \sigma_\varepsilon^2\right)\right)^2} < 0.$$

Hence, a drop in the ratio r_{n-f} increases the expected prices for both the market portfolio and individual assets.

[2] Glosten et al. (2016) point out that the increasing co-movement can be partly driven by fundamental factors, which lines up with this situation in my model. Whereas other traditional literature show a non-fundamental factor-driven results (e.g., Harris and Gurel, 1986; Vijh, 1994; Barberis et al., 2005; Peng and Xiong, 2006; Greenwood, 2007; Staer, 2014; Da and Shive, 2018).

(b) I can rewrite the coefficients with respect to the ratio r_{n-f}, as follows,

$$\theta_q = \gamma(1+r_{n-f})\,\sigma_\varepsilon^2,$$
$$\theta_b = \frac{\sigma_s^2\left(1+\gamma^2(1+r_{n-f})\,\sigma_{q_I}^2\,\sigma_\varepsilon^2\right)}{\sigma_\varepsilon^2+\gamma^2(1+r_{n-f})^2\,\sigma_{q_I}^2\,\sigma_\varepsilon^4+\sigma_s^2\left(1+\gamma^2(1+r_{n-f})\,\sigma_{q_I}^2\,\sigma_\varepsilon^2\right)}.$$

Notice that
$$\frac{\partial\theta_0}{\partial r_{n-f}} = \bar{q}_I\,\gamma\sigma_s^2\,\frac{\partial\theta_b}{\partial r_{n-f}}.$$

As r_{n-f} decreases, θ_q decreases and θ_b increases. Therefore, the effect on variance $Var(p_I)=\theta_b^2\,\sigma_s^2+\theta_b^2\,\theta_q^2\,\sigma_{q_I}^2$, at a first glance, turns out to be mixed. In fact, differentiating the variance with respect to r_{n-f} yields

$$2\gamma^2\,\sigma_{q_I}^2\,\sigma_s^4\,\sigma_\varepsilon^4\left[-r_{n-f}\,\sigma_s^2+(1+r_{n-f})(1+\gamma^2\,\sigma_{q_I}^2\,\sigma_s^2)\,\sigma_\varepsilon^2+\gamma^2(1+r_{n-f})^2\,\sigma_{q_I}^2\,(2-r_{n-f}+2\gamma^2(1+r_{n-f})\,\sigma_{q_I}^2\,\sigma_s^2)\,\sigma_\varepsilon^4\right.$$
$$\left.+\gamma^4(1+r_{n-f})^3\,\sigma_{q_I}^4\,(1-r_{n-f}+\gamma^2(1+r_{n-f})\,\sigma_{q_I}^2\,\sigma_s^2)\,\sigma_\varepsilon^6\right]/\left(\sigma_\varepsilon^2+\gamma^2(1+r_{n-f})^2\,\sigma_{q_I}^2\,\sigma_\varepsilon^4+\sigma_s^2\left(1+\gamma^2(1+r_{n-f})\,\sigma_{q_I}^2\,\sigma_\varepsilon^2\right)\right)^3.$$

The denominator is always positive. When the information is relatively precise, i.e., σ_ε^2 is tiny, the first term in the bracket of numerator dominates, as the rest are the higher orders of σ_ε^2. Thus in this case the variance of index price is negatively correlated with the ratio r_{n-f}.

(c) I calculate the covariance using the definition, i.e.,

$$Cov(p_2,p_3)=Cov(\tilde{p}_2+\beta_2\,p_I,\tilde{p}_3+\beta_3\,p_I)=\phi_a^2\,Cov(e_2,e_3)+\rho_{23}\,\phi_a^2\,\phi_q^2\,\sigma_{q_2}^2+\beta_2\,\beta_3\,Var(p_I).$$

If the index is traded actively enough, i.e., if $\sigma_{q_I}^2$ dominates, then $Cov(p_2,p_3)$ moves in the same way as $Var(p_I)$.

Proof of Lemma 9 Simply taking derivatives of η_I and $\tilde{\eta}_2$ over r_{f-f} yields the results. From the relation (C-7), one can see that η_2 moves in the same direction as $\tilde{\eta}_2$. Note that the effect of r_{f-f} on price efficiency is more sophisticated, rather than quadratic.

D Heterogeneous Households

D.1 Who favors factor investing?

Each household, with statistical structure $\{W^n, c^n, \gamma^n\}$, will employ the strategy

$$\tilde{\boldsymbol{x}}_h^n = \left(\frac{\mathbb{E}[S|p_I] - p_I}{\gamma^n\, Var(S|p_I)}, 0 \right)^{\mathsf{T}},$$

if she chooses to invest herself. The funds' strategies are independent of n and identical to all the households. However, because wealth endowment influences risk appetite, the strategies generate heterogeneous certainty-equivalent utility to heterogeneous households. Following the analysis in Lemma C-1, we have the following certainty-equivalent payoffs to household n from different investment vehicles, denoted by u_j^n $(j = a, p, h)$,

$$\begin{aligned}
u_a^n &= \frac{1}{2\gamma^n} \left[\log\left(\frac{Var(S - p_I)}{Var(S|s)} \right) + \log\left(\frac{Var(\mathscr{E}_2 - \tilde{p}_2)}{Var(\mathscr{E}_2|e_2)} \right) + \frac{(\bar{s} - \theta_0)^2}{Var(S - p_I)} \right], \\
u_b^n &= \frac{1}{2\gamma^n} \left[\log\left(\frac{Var(S - p_I)}{Var(S|s)} \right) + \frac{(\bar{s} - \theta_0)^2}{Var(S - p_I)} \right], \\
u_h^n &= \frac{1}{2\gamma^n} \left[\log\left(\frac{Var(S - p_I)}{Var(S|p_I)} \right) + \frac{(\bar{s} - \theta_0)^2}{Var(S - p_I)} \right].
\end{aligned}$$

Notice that $\gamma^n u_j^n$ is independent of n.

A household weakly prefers skilled active manager to investing herself if and only if

$$u_a^n - \alpha_a^n - c^n \geq u_h^n, \;\; \Rightarrow \;\; \gamma^n (u_a^n - \alpha_a^n - c^n) \geq \gamma^n u_h^n. \tag{D-1}$$

Because the fees charged by funds are proportional to a household's wealth, a household is indifferent between direct investing and factor investing. Thus one can figure out the equilibrium fees of factor funds, as follows,

$$u_b^n - \alpha_b^n = u_h^n, \;\; \Rightarrow \;\; \gamma^n \alpha_b^n = \gamma^n (u_b^n - u_h^n) = \eta_I, \;\; \Rightarrow \;\; \alpha_b^n = \frac{\eta_I}{\gamma^R} W^n.$$

Similar to relation (3.4), we have $(\alpha_a^n - \alpha_b^n)/W^n = k$, where k is the manager's information cost per unit of wealth. Then the equilibrium active delegation fees for household n become

$$\alpha_a^n = \left(\frac{\eta_I}{\gamma^R} + k \right) W^n.$$

Similarly, the model implies that households cannot gain from selecting active managers based on prior information, i.e.,

$$\mu\, e^{-\gamma^n (u_a^n - \alpha_a^n)} + (1-\mu)\, e^{-\gamma^n (u_b^n - \alpha_a^n)} = e^{-\gamma^n u_h^n},$$

which, performing necessary calculation, yields

$$e^{-\gamma^n (u_a^n - u_b^n - kW^n)} = \frac{1 - e^{\gamma^R k}(1-\mu)}{\mu}.$$

Linking the result to condition (D-1), (D-1) can be reduced to a feasible region determined by μ:

$$\gamma^n (u_a^n - u_b^n - kW^n) = -\log\left(\frac{1 - e^{\gamma^R k}(1-\mu)}{\mu} \right) \geq \gamma^R \frac{c^n}{W^n}. \tag{D-2}$$

Thus, households with either larger wealth or higher cognitive ability are more likely to choose fundamental investing. While the rest, termed unsophisticated households, stay uninformed and favor more conservative investment vehicles. Up till now I have proved condition (4.2) in Lemma 4.

D.2 Equilibrium price and price efficiency

Because the wealth distribution across the population of households is unequal, each individual could contribute differently to the funds inflow due to their wealth effect. Therefore, I first figure out the equilibrium effect on market clearing, asset price, and price efficiency. Once these terms are determined, they shall be shared by all the households. In other words, defining $\mathbb{E}[\frac{1}{\gamma^n}] = \frac{1}{\gamma}$, the equilibrium price and price efficiency are independent of n.

The market clearing condition with wealth effect becomes

$$\frac{1}{W}\Big(\lambda_h\,\mathbb{E}[W^n\,\tilde{\boldsymbol{x}}_h^n|\text{uninformed}] + \big(\lambda_b + (1-\mu)\,(1-\xi)\,\lambda_a\big)\,\mathbb{E}[W^n|\text{uninformed}]\,\tilde{\boldsymbol{x}}_b \\ + \big(\xi\,\mathbb{E}[W^n|\text{informed}] + \mu\,(1-\xi)\,\mathbb{E}[W^n|\text{uninformed}]\big)\,\lambda_a\,\tilde{\boldsymbol{x}}_a\Big) = \tilde{\boldsymbol{q}},$$

scaling for expected wealth. Correspondingly, the coefficients of equilibrium price become

$$\theta_q = \frac{\gamma^R\,\sigma_\varepsilon^2}{\lambda_b\,\mathbb{E}[W^n|\text{uninformed}] + \lambda_a\,W},$$
$$\phi_q = \frac{\gamma^R\,\sigma_\varepsilon^2}{\big(\xi\,\mathbb{E}[W^n|\text{informed}] + \mu\,(1-\xi)\,\mathbb{E}[W^n|\text{uninformed}]\big)\,\lambda_a}.$$

Recall the definition $\beta_{v,s} = \frac{\sigma_s^2}{\sigma_s^2+\sigma_\varepsilon^2}$ and $\beta_{v,e} = \frac{\sigma_e^2}{\sigma_e^2+\sigma_\varepsilon^2}$. Hence, the price inefficiency of market portfolio, η_I, becomes

$$\eta_I = -\frac{1}{2}\,\log\left(1 - \frac{\beta_{v,s}\,\sigma_\varepsilon^2\,\sigma_{q_I}^2}{\big((\lambda_b\,\mathbb{E}[W^n|\text{uninformed}] + \lambda_a\,W)/\gamma^R\big)^2 + \sigma_\varepsilon^2\,\sigma_{q_I}^2}\right). \tag{D-3}$$

Following the same rationale in Lemma C-1, one can characterize the relative utility in the heterogeneous world. For factor funds, we have

$$\gamma^n\,(u_b^n - u_h^n) = \eta_I.$$

For fundamental investors, we have

$$\gamma^n\,(u_a^n - u_h^n) = \eta_I + \frac{1}{2}\,\log\left(1 + \frac{(\gamma^R)^2\,\beta_{v,e}\,\sigma_\varepsilon^2\,\sigma_{q_2}^2}{\big(\xi\,\mathbb{E}[W^n|\text{informed}] + \mu\,(1-\xi)\,\mathbb{E}[W^n|\text{uninformed}]\big)^2\,\lambda_a^2}\right). \tag{D-4}$$

D.3 Asset market equilibrium

The equilibrium amount of informed households, ξ, is determined endogenously given condition (D-2):

$$\xi = \mathbb{P}(u_a^n - c^n - \alpha_a^n \geq u_h^n) = \mathbb{P}\left(\gamma^n\,(u_a^n - u_b^n - k\,W^n) \geq \gamma^R\,\frac{c^n}{W^n}\right) = \mathbb{P}\left(c^n \leq -\frac{1}{\gamma^R}\,\log\Big(\frac{1 - e^{\gamma^R k}\,(1-\mu)}{\mu}\Big)\,W^n\right).$$

Given the distribution of $\{W^n, c^n\}$, with a joint density function $f_{W^n,c^n}(\cdot,\cdot)$ (separable if the two variables are independent), the probability ξ is related to the region determined by μ, which is another endogenous value. Denote the region created by condition (D-2) and the support of distribution $f_{W^n,c^n}(\cdot,\cdot)$ as $\Omega(\mu)$. Then mathematically,

$$\xi = \mathbb{P}\big((W^n, c^n) \in \Omega(\mu)\big) = \iint_{\Omega(\mu)} f_{W^n,c^n}(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

which implies that ξ is a function of μ.

Now I turn to the fund side. First, the skilled managers attract all the inflows from informed households and part of the inflows from the uninformed ones. Whereas the closet indexers can only gain by fooling the uninformed households. Thus the coexistence of skilled and opportunistic managers implies

$$\left(\mathbb{E}[\alpha_a^n - k\,W^n|\text{informed}]\,\frac{\xi}{\mu} + \mathbb{E}[\alpha_a^n - k\,W^n|\text{uninformed}]\,\frac{\mu\,(1-\xi)}{\mu}\right)\lambda_a = \mathbb{E}[\alpha_a^n\,W^n|\text{uninformed}]\,\frac{(1-\mu)\,(1-\xi)}{1-\mu}\,\lambda_a,$$

which, after simplification, yields

$$\frac{\eta_I}{\gamma^R \mu} \mathbb{E}[W^n|\text{informed}]\,\xi = k\,\mathbb{E}[W^n|\text{uninformed}]\,(1-\xi). \tag{D-5}$$

Note that

$$\mathbb{E}[W^n|\text{informed}]\,\xi = \mathbb{E}\big[W^n|(W^n,c^n)\in\Omega(\mu)\big]\,\mathbb{P}\big((W^n,c^n)\in\Omega(\mu)\big) = \iint_{\Omega(\mu)} x\, f_{W^n,c^n}(x,y)\,\mathrm{d}x\mathrm{d}y,$$

and similar for $\mathbb{E}[W^n|\text{uninformed}]\,(1-\xi)$. Then through finding the fixed point of equation (D-5), I express μ as a function of η_I. The household investment in finding out managers, ξ, determined by μ, becomes a function of η_I as well.

Second, the coexistence of factor and active funds shows that

$$\mathbb{E}[\alpha_b^n W^n|\text{uninformed}]\,\lambda_b = \mathbb{E}[\alpha_a^n W^n|\text{uninformed}]\,\frac{(1-\mu)(1-\xi)}{1-\mu}\,\lambda_a,$$

which implies

$$\frac{\lambda_b}{\lambda_a} = \frac{\eta_I+\gamma^R k}{\eta_I}\,(1-\xi).$$

Recall the form of η_I given by equation (D-3). With the above ratio of capital in factor over fundamental investing, one can express λ_a using η_I, as follows,

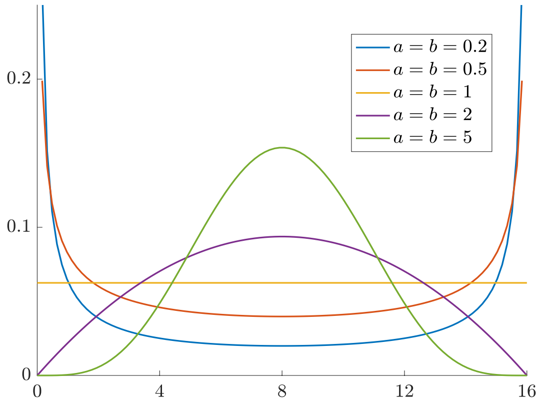
$$W\lambda_a + \mathbb{E}[W^n|\text{uninformed}]\,\lambda_b = \left(\frac{\eta_I+\gamma^R k}{\eta_I}(1-\xi)\,\mathbb{E}[W^n|\text{uninformed}]+W\right)\lambda_a = \gamma^R\sigma_\varepsilon\sigma_{q_I}\sqrt{\frac{\beta_{v,s}}{1-e^{-2\eta_I}}-1}.$$

Finally, I rely on the result of relative utility, equation (D-4), to solve the fixed point η_I. Similar to equation (C-14) in the proof of Proposition 2, and combining condition (D-2), we have

$$\gamma^n(u_a^n-u_b^n) = \frac{1}{2}\log\left(1+\frac{(\gamma^R)^2\beta_{v,e}\sigma_\varepsilon^2\sigma_{q_2}^2}{\big(\xi\,\mathbb{E}[W^n|\text{informed}]+\mu(1-\xi)\,\mathbb{E}[W^n|\text{uninformed}]\big)^2\lambda_a^2}\right) = \gamma^R k - \log\left(\frac{1-e^{\gamma^R k}(1-\mu)}{\mu}\right),$$

where μ, ξ, and λ_a are all functions of η_I. Therefore, one can determine the equilibrium price informativeness, then the rest terms follow.

Lemma 5 Before showing the result, I first give the details of how to measure wealth dispersion. As shown in the main body, I vary parameter a of beta distribution, Beta(a,a), holding the mean and support constant. A higher value of a implies a more concentrated statistical distribution, intuitively, shown in Figure D-4. Taking Gini coefficient as a measure of wealth dispersion, a higher a maps into a lower Gini coefficient.



a = b	Gini
0.2	0.47
0.5	0.41
1	0.33
2	0.26
5	0.17

Figure D-4: *Density of Beta distribution, Beta(a,b), for various parameters $a=b$, and the corresponding Gini index.*

Fixing the information cost c, the condition to determine ξ reduces to

$$\xi = \mathbb{P}\left(W^n \geq -\frac{\gamma^R c}{\log\left(\frac{1-e^{\gamma^R k}(1-\mu)}{\mu}\right)} \right).$$

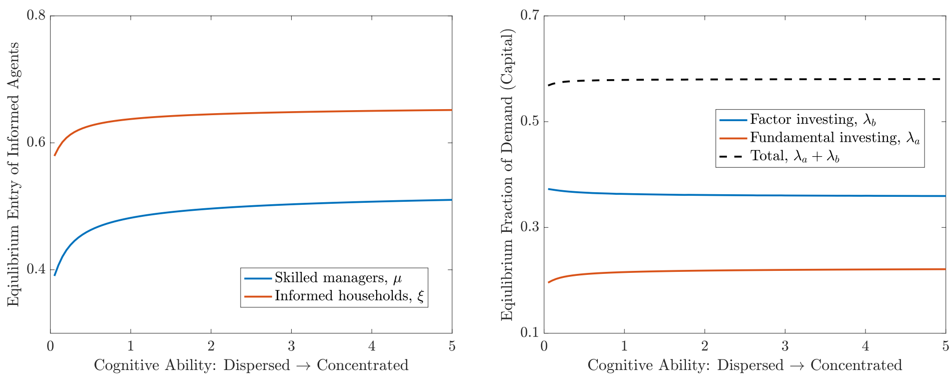
The rest of the analysis hold. Thus numerically, as I vary the choice of parameter a of Beta distribution to control wealth dispersion level, the asset market equilibrium terms vary correspondingly. An intuitive example and explanation are provided in the main body of the paper. It shows that higher wealth inequality engenders more skilled managers, less households willing to become informed, and higher capital in both funds. The outcomes are robust across different choices of exogenous variables.

Lemma 6 Lemma 6 follows exactly the same rationale. With a fixed wealth across the population of households $W^n \equiv W$, households investment in finding out manager quality, ξ, is determined by

$$\xi = \mathbb{P}\left(c^n \leq -\frac{W}{\gamma^R} \log\left(\frac{1-e^{\gamma^R k}(1-\mu)}{\mu} \right) \right) = \frac{B\left(-\frac{W}{\gamma^R}\log\left(\frac{1-e^{\gamma^R k}(1-\mu)}{\mu}\right); a, a\right)}{B(a,a)},$$

where $B(\cdot\,; a, a)$ is the incomplete beta function and $B(a,a)$ is the beta function.

Cognitive ability, in this framework, does not affect certainty-equivalent wealth. Moreover, fixing the wealth level, households' contribution towards fund inflow is identical. Therefore I do not need to distinguish between demand and capital. A numerical example is given below. Through examining a series of reasonable choice of exogenous variables, one can tell that the effect is robust, yet of a tiny quantity.



A. Informed agents.

B. Demand/Capital.

Figure D-5: *Cognitive ability inequality on capital allocation.* I vary the parameter of Beta distribution, a, in $(0,5]$, holding the expected information cost $c = 0.3$ and its support $[0, 0.6]$. For the other parameters, take $\gamma^R = 3$, $k = 0.05$, $W = 5$, $\sigma_{q_1}^2 = \sigma_{q_2}^2 = \sigma_e^2 = \sigma_s^2 = 2\sigma_\varepsilon^2 = 0.6$.

Lemma 7 The density of AMH copula, by calculation, for any given correlation parameter θ, is

$$f_\theta(x,y) = \frac{\partial \text{AMH}(\theta)}{\partial x \partial y} = \frac{4Wc\left(2c(1-\theta)(2W(1-\theta)+x\theta)+y\theta(2W+x-2W\theta+x\theta)\right)}{\left(4Wc-\theta(2W-x)(2c-y)\right)^3}.$$

Recall the boundary condition to classify sophisticated and unsophisticated households, equation (4.2),

$$c^n \leq -\frac{1}{\gamma^R}\log\left(\frac{1-e^{\gamma^R k}(1-\mu)}{\mu}\right)W^n.$$

The region created by this condition, Ω, is determined by the line crossing origin with slope $\delta := -\frac{1}{\gamma^R}\log(\frac{1-e^{\gamma^R k}(1-\mu)}{\mu})$, and the projected rectangular domain $(W^n, c^n) \in [0, 2W] \times [0, 2c]$. If the slope is weakly smaller than c/W, then the region is of a triangle shape, with ξ determined by

$$\xi = \iint_{\Omega(\mu)} f_\theta(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2W}\int_0^{\delta x} f_\theta(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

and

$$\mathbb{E}[W^n|\text{informed}]\,\xi = \iint_{\Omega(\mu)} x\,f_\theta(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2W}\int_0^{\delta x} x\,f_\theta(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$
$$\mathbb{E}[W^n|\text{uninformed}]\,(1-\xi) = W - \mathbb{E}[W^n|\text{informed}]\,\xi.$$

If the slope is larger than c/W, then the region is of a trapezoid shape, with ξ determined by

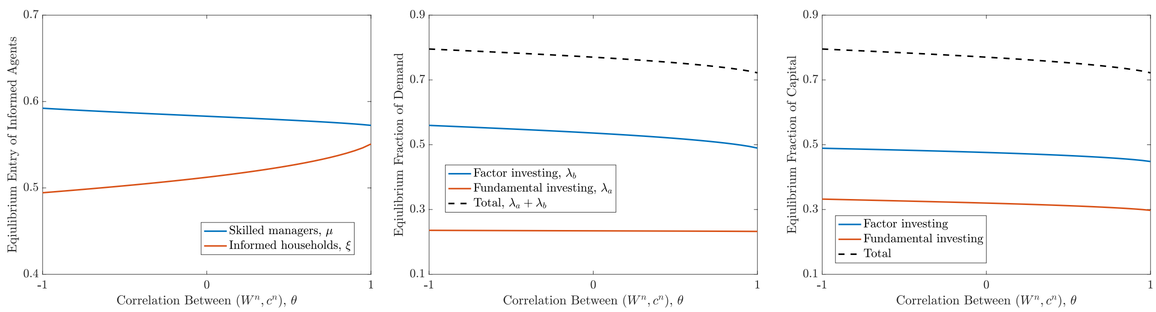
$$\xi = 1 - \int_0^{2c}\int_0^{\frac{y}{\delta}} f_\theta(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

and

$$\mathbb{E}[W^n|\text{uninformed}]\,(1-\xi) = \int_0^{2c}\int_0^{\frac{y}{\delta}} x\,f_\theta(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$
$$\mathbb{E}[W^n|\text{informed}]\,\xi = W - \mathbb{E}[W^n|\text{uninformed}]\,(1-\xi).$$

Thus one can solve μ according to condition (D-5) as a function of η_I, and solve η_I correspondingly following the above analysis.

Numerically, the evidence that the correlation between wealth and cognitive ability influences the equilibrium capital allocation is given in Figure D-6. The outcomes are robust across different choices of exogenous variables: A negative θ implies that a wealthier household may have more insight towards a manager's quality. An unequal society stimulates demand for factor investing (Panel B and C of Figure D-6), as well as increases the competitive pressure of active counterparts so that they have to lower their fees. The results are very much similar to the ones delivered by wealth inequality, which is reasonable because wealth effect dominates cognitive ability.



A. Informed agents. B. Demand. C. Capital.

Figure D-6: *Unequal society on capital allocation.* I vary the parameter of AMH copula, θ, in $[-1,1]$, holding the expected wealth $W=5$ and information cost $c=0.2$. For the other parameters, take $\gamma^R=3$, $k=0.05$, $\sigma_{q_I}^2=\sigma_{q_2}^2=\sigma_e^2=\sigma_s^2=2\sigma_\varepsilon^2=0.5$.

D.4 Dynamics of opportunism managers

Proof of Example 1 I mainly derive the results for the heterogeneous equilibrium ("talent" market). I always use overscore for larger values and underscore for smaller values in notations. For the wealthy and informed households, with mass ξ, their properties and certainty-equivalent utilities can be characterized by the following set,

$$\underline{c},\ \overline{W},\ \underline{\gamma}=\frac{\gamma^R}{\overline{W}},\ \overline{u}_a,\ \overline{u}_b,\ \overline{u}_h,\ \overline{\alpha}_b=\frac{\eta_I}{\gamma^R}\overline{W},\ \overline{\alpha}_a=\left(\frac{\eta_I}{\gamma^R}+k\right)\overline{W}.$$

For the rest uninformed households, the corresponding set becomes

$$\overline{c},\ \underline{W},\ \overline{\gamma}=\frac{\gamma^R}{\underline{W}},\ \underline{u}_a,\ \underline{u}_b,\ \underline{u}_h,\ \underline{\alpha}_b=\frac{\eta_I}{\gamma^R}\,\underline{W},\ \underline{\alpha}_a=\Big(\frac{\eta_I}{\gamma^R}+k\Big)\,\underline{W}.$$

Note that we have $\overline{\gamma}\underline{u}_j=\underline{\gamma}\overline{u}_j$. The average absolute risk aversion level is determined by $\gamma=\frac{\gamma^R}{\mathbb{E}[W^n]}=\frac{\gamma^R}{\xi\,\overline{W}+(1-\xi)\,\underline{W}}$.

The rich and insightful group of households, as shown in the analysis in the main body, strictly prefers skilled fundamental investors, mathematically,

$$\overline{u}_a-\overline{\alpha}_a-\underline{c}>\overline{u}_b-\overline{\alpha}_b=\overline{u}_h.$$

Correspondingly, for the other group of households, we have

$$\underline{u}_a-\underline{\alpha}_a-\overline{c}<\underline{u}_b-\underline{\alpha}_b=\underline{u}_h,$$
$$\mu\,e^{-\overline{\gamma}(\underline{u}_a-\underline{\alpha}_a)}+(1-\mu)\,e^{-\overline{\gamma}(\underline{u}_b-\underline{\alpha}_a)}=e^{-\overline{\gamma}\underline{u}_h}.$$

Managers self-select into their roles, as before. However, the contribution of the two groups of households to the fund inflows is different due to their wealth effect, yielding

$$\Big((\overline{\alpha}_a-k\overline{W})\,\frac{\xi}{\mu}+(\underline{\alpha}_a-k\underline{W})\,\frac{\mu\,(1-\xi)}{\mu}\Big)\,\lambda_a=\underline{\alpha}_a\,\frac{(1-\mu)\,(1-\xi)}{1-\mu}\,\lambda_a.$$
$$\underline{\alpha}_b\,\lambda_b=\underline{\alpha}_a\,(1-\xi)\,\lambda_a.$$

Rearranging the definition of price efficiency yields

$$\lambda_a\,W+\lambda_b\,\underline{W}=\gamma^R\,\sigma_\varepsilon\,\sigma_{q_I}\sqrt{\frac{\beta_{v,s}}{1-e^{-2\,\eta_I}}-1}.$$

Combine it with the funds coexisting conditions, I can express μ and λ_a both as functions of η_I:

$$\mu=\frac{\eta_I}{k\,\gamma^R}\,\frac{\overline{W}\,\xi}{\underline{W}\,(1-\xi)},$$
$$\lambda_a=\frac{\gamma^R\,\sigma_\varepsilon\,\sigma_{q_I}\,\eta_I}{(\eta_I+k\,\gamma^R)\,(1-\xi)\,\underline{W}+\eta_I\,W}\sqrt{\frac{\beta_{v,s}}{1-e^{-2\,\eta_I}}-1}.$$

Finally, I rely on the result of relative utility, Lemma C-1, to solve the fixed point η_I:

$$\overline{\gamma}(\underline{u}_a-\underline{u}_b)=\frac{1}{2}\,\log\Big(1+\frac{(\gamma^R)^2\,\beta_{v,e}\,\sigma_\varepsilon^2}{(\xi\,\overline{W}+\mu\,(1-\xi)\,\underline{W})^2\,\lambda_a^2}\,\sigma_{q_2}^2\Big)=\gamma^R\,k-\log\Big(\frac{1-e^{\gamma^R k}\,(1-\mu)}{\mu}\Big),$$

Then performing the same calculation as before yields the equilibrium with heterogeneous households.

For the homogeneous model, all the previous analysis applies. The only difference is due to the set up of non-unit wealth level, $W=2$. Then condition (3.4) becomes

$$(\alpha_a-\alpha_b)/W=k,$$

i.e., fees are proportional to the total wealth.

E Rich Asset Market

For tractability I analyze the asset market equilibrium assuming $l = 2$ in the main body of the paper. However the rationale is applicable to any asset market with more than two underlying assets. Specially, in the real financial market, the indices usually consist of more than 100 stocks, for instance, S&P 500, Russell 2000, FTSE 100, etc. Therefore, I extend my analysis to cover the case where asset market is "rich" enough, i.e., l is large. In this appendix I basically show that all the analysis in the main body holds for a world with large number of assets, and use the model to bridge manager skill and CAPM.

Assume index asset has a composition of $b_i = 1/l$ for any $i \in \{1,2,...,l\}$. As the index averages out idiosyncratic risks, one feasible covariance matrix, symmetric and positive-definite, features an identical correlation between any two idiosyncratic loadings:

$$\rho = \frac{Cov(\mathscr{E}_i, \mathscr{E}_j)}{\sigma_e^2} = -\frac{1}{l-1}, \qquad \forall i \neq j.$$

For an l large enough, this approximately generates independency among idiosyncratic risks. Thus the underlying assets are only correlated in terms of their factor loadings.

E.1 Equilibrium prices

Because a large l implies $\rho \to 0$, we have approximately the following equilibrium asset prices.

Corollary E.1 (Equilibrium prices) *The index price remains the same as in Proposition 1:*

$$p_I = \theta_0 + \theta_b \left[(s - \bar{s}) - \theta_q (q_I - \bar{q}_I) \right];$$

whereas the individual asset price satisfies, approximately, for any $i \in \{1,2,...,l\}$,

$$p_i = \beta_i \, p_I + \phi_a \, (e_i - \phi_q \, q_i).$$

where q_i is the supply of $\mathscr{E}_i$, and $(\theta_0, \theta_b, \theta_q)$ and (ϕ_a, ϕ_q) are parameters determined in the proof.

With the equilibrium prices, I am able to recover the price efficiency as before. Given the capital delegated to funds, the index price contains the same information as given in equation (2.6). Provided that $\sigma_{q_i}^2 \equiv \sigma_q^2$, the price informativeness of idiosyncratic attributes, equation (C-6), can be generalized to

$$\tilde{\eta}_i := -\frac{1}{2} \log \Big(\frac{Var(\mathscr{E}_i | e_i)}{Var(\mathscr{E}_i | \tilde{p}_i)} \Big) = -\frac{1}{2} \log \left(1 - \frac{\sigma_\varepsilon^2 \, \sigma_e^2 \, \sigma_q^2 / (\sigma_e^2 + \sigma_\varepsilon^2)}{(\, \xi + \mu \, (1 - \xi) \,)^2 \, \lambda_a^2 / \gamma^2 + \sigma_\varepsilon^2 \, \sigma_q^2} \right) \in (0, \infty), \qquad \forall i.$$

To make a reasonable assumption, let fundamental investors be able to produce idiosyncratic information of m stocks ($m \leq n$). In a market with large number of underlying assets, it is fair that factor investors have weak signal of, say, 500 stocks thus are able to time the factor, while fundamental investors have strong signal towards 20 attributes of the individual assets in the basket. A proper choice of m captures the rationale in a stylized fashion. Following the same rationale, I relate price efficiency to relative utilities, with the approximate results specified in Corollary E.2.

Corollary E.2 (Relative utility) *The excess performance of fund industry, represented using the relative utility, can be linked to the price efficiency in the following ways:*

$$\gamma(u_b - u_h) = \eta_I,$$

$$\gamma(u_a - u_b) = \frac{m-1}{2} \log \Big(1 + \frac{\gamma^2 \, \sigma_\varepsilon^2 \, \sigma_e^2 \, \sigma_q^2 / (\sigma_e^2 + \sigma_\varepsilon^2)}{(\, \xi + \mu \, (1 - \xi) \,)^2 \, \lambda_a^2} \Big).$$

E.2 Asset market equilibrium

Now I discuss how a rich asset market affects the equilibrium terms. Recall the delegation fees charged by active managers, equation (3.4). It is fair that the information cost of active managers increases with the number of assets they emphasize on, m, yet is bounded from above as the marginal cost of investigating the final asset is approximately zero. Thus I update equation (3.4) to be $\alpha_a - \alpha_b = k(m)$. Provided with the previous analysis, I assume $k(m)$ is Lipschitz continuous.

Then with the main ideas remaining unchanged, I find that a relatively large m can increase the amount of factor investing, in fact, fundamental investing as well. Mathematically we have

$$\lambda_b = \frac{\gamma \sigma_{q_2}}{\mu} \sqrt{\frac{1}{\left(\frac{1}{\sigma_\epsilon^2} + \frac{1}{\sigma_\varepsilon^2}\right)\left(e^{\frac{2\gamma(k(m)+c)}{m-1}} - 1\right)}}.$$

When there are many underlying assets, another outcome is the increasing index price efficiency. This follows directly from the increase in money invested in funds—the more informed trading conducted by the fund managers, the more information the final price reflects. Furthermore, though the entry of skilled managers, μ, is unrelated to the number of assets, the amount of informed households, ξ, increases due to the higher price efficiency. On the one hand, with more informed households delegating wealth to skilled active managers, each individual asset contributes less excess return to the selected portfolio. On the other hand, more idiosyncratic loadings provide extra space for fundamental investors to profit. Therefore the negative and positive effects partly cancel out with each other. Finally, because the coexistence of factor and active funds always implies that $\left(\xi + \mu\,(1-\xi)\right)\lambda_a = \mu\,\lambda_b$ in an interior equilibrium, one can conclude that the price efficiency of individual assets also enhances.

All the rest analysis of equilibria remains unchanged. There still exists an upper bound of the fraction of capital in factor investing, yet the condition upon reaching the peak level varies with the number of assets.

E.3 Skilled active funds and alpha

I now discuss the outperformance of skilled active managers. One common way to estimate a fund's excess return is through the Capital Asset Pricing Model (CAPM) and look into the "alpha," i.e., the intercept of the Security Market Line (SML), that fund generates. The only group of investors who can capture the abnormal return (funds' return minus the market return) is the group of fundamental investors, who hold inside information of idiosyncratic loadings. They are known for charging higher fees due to the alpha they generate.

Recalling the results in Section 2.3, skilled active managers' strategy is $\boldsymbol{x}_a = (x_{1a}, x_{2a}, ..., x_{la})^\intercal$, specified by equation (2.4). Then the weight put on asset i by the skilled manager is

$$\omega_{ia} = \frac{x_{ia}\, p_i}{\boldsymbol{x}_a^\intercal \boldsymbol{p}}, \qquad i = 1, 2, ..., l.$$

The gross returns of asset i, R_i, and market portfolio, R_m, are

$$R_i = \frac{v_i}{p_i}, \qquad R_m = \frac{S}{p_I}, \qquad i = 1, 2, ..., l.$$

Define the vector of risky asset weight $\boldsymbol{\omega}_a = (\omega_{1a}, \omega_{2a}, ..., \omega_{la})^\intercal$ and the vector of risky asset return $\boldsymbol{R} = (R_1, R_2, ..., R_l)^\intercal$. Then the skilled funds' return, denoted by R_a, can be expressed as $R_a = \boldsymbol{\omega}_a^\intercal \boldsymbol{R}$. Because the risk-free rate is normalized to be one, I use $\mathbf{1}$ as the l-dimensional unit vector. In the following lemma I develop the model-implied CAPM for the expected excess fund return.

Lemma E-1 (**Model-implied CAPM**) *If the supply of systematic risk dominates, then the expected excess return of skilled active funds satisfies*

$$\mathbb{E}[R_a] - 1 = \mathbb{E}[\boldsymbol{\omega}_a^\intercal (\boldsymbol{R} - 1)] = \alpha + \beta\,(\mathbb{E}[R_m] - 1).$$

The intercept satisfies $\alpha = \sum_{i=1}^{l} \frac{1}{\gamma}\left(\frac{Var(\tilde{v}_i - \tilde{p}_i)}{Var(\tilde{v}_i | \mathscr{I}_a)} - 1\right) - A_i$, *where* $\tilde{v}_i$ *is the payoff of the orthogonal asset* i *defined* *in Lemma B-1 and $\tilde{p}_i$ is the corresponding price. The slope satisfies $\beta = \sum_{i=1}^{l} \hat{\omega}_{ia} \frac{Cov(R_i,R_m)}{Var(R_m)}$. A_i and $\hat{\omega}_{ia}$ are positive numbers and will be specified in the proof.*

PROOF: The weight placed on asset i is defined to be $\omega_{ia} = \frac{x_{ia}\, p_i}{\boldsymbol{x}_a^\intercal \boldsymbol{p}}$. Note that the element will simply be zero if fundamental investors do not hold this asset. Thus the expected excess return satisfies

$$\begin{aligned}\mathbb{E}[R_a - 1] &= \mathbb{E}[\,\boldsymbol{\omega}_a^\intercal (\boldsymbol{R} - 1)\,] = \sum_{i=1}^{l} \mathbb{E}\Big[\,\frac{x_{ia}\, p_i}{\boldsymbol{x}_a^\intercal \boldsymbol{p}} \Big(\frac{v_i}{p_i} - 1\Big)\Big] = \frac{1}{\boldsymbol{x}_a^\intercal \boldsymbol{p}} \sum_{i=1}^{l} \mathbb{E}[\,x_{ia}\,(v_i - p_i)\,] \\ &= \frac{1}{\boldsymbol{x}_a^\intercal \boldsymbol{p}} \sum_{i=1}^{l} \Big(\,\mathbb{E}[x_{ia}]\,\mathbb{E}[v_i - p_i] + Cov(x_{ia}, v_i - p_i)\,\Big).\end{aligned} \tag{E-1}$$

First, I work out the second term on the RHS—the sum of covariances. In matrix notation, $\sum_{i=1}^{n} Cov(x_{ia}, v_i - p_i) = \mathrm{Tr}(\,Cov(\boldsymbol{x}_a, \boldsymbol{v} - \boldsymbol{p})\,)$, where 'Tr' represents the trace of a matrix. If, by orthogonalization, I relate the payoff vector in original and orthogonal asset space, $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$, through a nonsingular transforming matrix Γ, i.e., $\boldsymbol{v} = \Gamma\,\tilde{\boldsymbol{v}}$, then we have

$$\mathrm{Tr}(\,Cov(\boldsymbol{x}_a, \boldsymbol{v} - \boldsymbol{p})\,) = \mathrm{Tr}(\,Cov(\,(\Gamma^{-1})'\tilde{\boldsymbol{x}}_a, \Gamma\,(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{p}})\,)\,) = (\Gamma^{-1}\Gamma)\,\mathrm{Tr}(\,Cov(\tilde{\boldsymbol{x}}_a, \tilde{\boldsymbol{v}} - \tilde{\boldsymbol{p}})\,).$$

Thus the sum of covariances can be simplified to

$$\sum_{i=1}^{l} Cov(x_{ia}, v_i - p_i) = \mathrm{Tr}(\,Cov(\tilde{\boldsymbol{x}}_a, \tilde{\boldsymbol{v}} - \tilde{\boldsymbol{p}})\,) = \sum_{i=1}^{m} Cov(\tilde{x}_{ia}, \tilde{v}_i - \tilde{p}_i) = \sum_{i=1}^{m} \frac{1}{\gamma}\,\frac{Var(\mathbb{E}[\tilde{v}_i - \tilde{p}_i|\mathscr{I}_a])}{Var(\tilde{v}_i|\mathscr{I}_a)} = \sum_{i=1}^{m} \frac{1}{\gamma}\Big(\frac{Var(\tilde{v}_i - \tilde{p}_i)}{Var(\tilde{v}_i|\mathscr{I}_a)} - 1\Big),$$

where the result lies on the strategy $\tilde{x}_{ia} = \frac{\mathbb{E}[\tilde{v}_i - \tilde{p}_i|\mathscr{I}_a]}{\gamma Var(\tilde{v}_i|\mathscr{I}_a)}$ and the law of total variance (C-8).

Next, consider the first term, i.e., the product of expectations. We have

$$\begin{aligned}\mathbb{E}[x_{ia}]\,\mathbb{E}[v_i - p_i] &= \mathbb{E}[x_{ia}]\,\mathbb{E}[p_i\,(R_i - 1)] \\ &= \mathbb{E}[x_{ia}]\,\big(\mathbb{E}[p_i]\,\mathbb{E}[R_i - 1] + Cov(p_i, R_i)\big) \\ &= \frac{Cov(R_i, R_m)}{Var(R_m)}\,\mathbb{E}[x_{ia}]\,\mathbb{E}[p_i]\,(\,\mathbb{E}[R_m] - 1\,) + \mathbb{E}[x_{ia}]\,Cov(p_i, R_i),\end{aligned}$$

where the last line holds approximately if the supply of index asset is relatively large, from $\mathbb{E}[R_i - 1] = \frac{Cov(R_i,R_m)}{Var(R_m)}\,(\,\mathbb{E}[R_m] - 1\,)$. The assumption is reasonable in that people usually regard index investment as where the liquidity is. Define $\hat{\omega}_{ia} = \frac{\mathbb{E}[x_{ia}]\,\mathbb{E}[p_i]}{\boldsymbol{x}_a^\intercal \boldsymbol{p}}$ and $A_i = \frac{\mathbb{E}[x_{ia}]\,Cov(p_i,R_i)}{\boldsymbol{x}_a^\intercal \boldsymbol{p}}$. Note that A_i is negative in that the price is negatively correlated with return ($Cov(p_i, R_i) < 0$).

Plugging the above results into expression of expected excess return, equation (E-1), then α and β follow.

Based on the results in Proposition 1, one has

$$Var(S|s) = \beta_{v,s}\,\sigma_\varepsilon^2,$$

and

$$Var(S - p_I) = (1 - \theta_b)^2\,\sigma_s^2 + \theta_b^2\,\sigma_\varepsilon^2 + \theta_b^2\,\theta_q^2\,\sigma_{p_I}^2.$$

The other return volatility and conditional variance terms follow similarly, especially when l is large, approximately,

$$Var(\tilde{v}_i|\mathscr{I}_a) = Var(\mathscr{E}_i|\mathscr{I}_a) = \beta_{v,e}\,\sigma_\varepsilon^2,$$

and

$$Var(\tilde{v}_i - \tilde{p}_i) = Var(\mathscr{E}_i - \tilde{p}_i) = \beta_{v,e}\,\sigma_\varepsilon^2\,\Big(\,1 + \frac{\gamma^2\,\beta_{v,e}\,\sigma_\varepsilon^2}{(\,\xi + \mu\,(1 - \xi)\,)^2\,\lambda_a^2}\,\sigma_q^2\Big).$$

Therefore, in any equilibrium, α is endogenously determined by the capital allocation strategy and investment in information acquisition.

As an example, take a look at the case when delegation dominates, i.e., $\lambda_h = 0$. This is essential as in a rich asset market, I have already shown that the fraction of households using delegation greatly increases. It is highly likely that direct investment can be completely driven out of the market. When delegation dominates, the following ratio could be simplified,

$$\frac{Var(S-p_I)}{Var(S|s)} = 1 + \gamma^2 \beta_{v,s} \sigma_\varepsilon^2 \sigma_{q_I}^2.$$

Then the abnormal return α becomes

$$\alpha = \gamma \sigma_\varepsilon^2 \left(\beta_{v,s} \sigma_{q_I}^2 + \frac{(m-1)\beta_{v,e} \sigma_q^2}{(\xi + \mu(1-\xi))^2 \lambda_a^2} \right) - \sum_{i=1}^{l} A_i.$$

□

As can be seen from the proof, when the systematic (σ_s^2) or idiosyncratic risk (σ_e^2) increases, α increases. The abnormal return is also larger when the supply of assets becomes more volatile.

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